Invasion by rare mutants in a spatial two-type Fisher-Wright system with selection

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Abstract

We consider a meanfield system of interacting Fisher-Wright diffusions with selection and rare mutation on the geographic space $\{1, 2, \cdots, N\}$. The type 1 has fitness 0, type 2 has fitness 1 and (rare) mutation occurs from type 1 to 2 at rate $m \cdot N^{-1}$, selection is at rate $s > 0$. The system starts in the state concentrated on type 1, the state of low fitness. We investigate this system for $N \to \infty$ on the original and large time scales.

We show that for some $\alpha \in (0, s)$ at times $\alpha^{-1} \log N + t, t \in \mathbb{R}, N \to \infty$ the emergence of type 2 (positive global type-2 intensity) at a global level occurs, while at times $\alpha^{-1} \log N + t_N$, with $t_N \to \infty$ we get fixation on type 2 and on the other hand with $t_N \to -\infty$ as $N \to \infty$ asymptotically only type 1 is present.

We describe the transition from emergence to fixation in the time scale $\alpha^{-1} \log N + t, t \in \mathbb{R}$ in the limit $N \to \infty$ by a McKean-Vlasov random entrance law. This entrance law behaves for $t \to -\infty$ like ${}^*\mathcal{W} e^{-\alpha|t|}$ for a positive random variable ${}^*\mathcal{W}$. The formation of small droplets of type-2 dominated sites in times $o(\log N)$, or $\gamma \cdot \log N, \gamma \in (0, \alpha^{-1})$ is described in the limit $N \to \infty$ by a measure-valued process following a stochastic equation driven by Poissonian type noise which we identify explicitly. The total mass of this limiting $(N \to \infty)$ droplet process grows like $\mathcal{W}^* e^{\alpha t}$ as $t \to \infty$. We prove that exit behaviour from the small time scale equals the entrance behaviour in the large time scale, namely $\mathcal{L}[{}^*\mathcal{W}] = \mathcal{L}[\mathcal{W}^*]$.

Keywords: Interacting Fisher-Wright diffusions, mutation, selection, rare mutation, punctuated equilibrium, random McKean-Vlasov equation, random entrance law, droplet formation, atomic measure-valued processes.

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1 The model and the results

1.1 Background and motivation

Mutation and selection both play essential roles in the evolution of a population. Mutation increases genetic diversity, while selection reduces this diversity pushing it towards states concentrated on the fittest types. The balance between these two forces is influenced heavily by the effect of migration. Randomness enters through resampling (pure genetic drift).

To give a precise setting for this study we model the population dynamics according to a stochastic process arising as the diffusion limit of many individuals incorporating the basic mechanisms for a spatial population traditionally employed in genetics and evolutionary biology. In this model *random fluctuations* are included in contrast to other infinite population limits which are ruled by deterministic differential equations (for example, see [Bu]). More precisely the model we use arises from the particle model driven by migration of particles between colonies, and in each site by resampling of types, mutation and selection. Increase the number of particles as ε^{-1} and give them mass ε. As $\varepsilon \to 0$, a diffusion limit of interacting multitype diffusions results if the resampling rate is proportional to the number of pairs and there is weak selection, i.e. selection occurs at a rate decreasing with the inverse of the number of particles per site. Otherwise with strong selection, i.e. selection at a fixed rate, we get the deterministic limit, often referred to as infinite population limit.

These stochastic models which describe populations of possibly infinitely many types are called interacting Fleming-Viot processes and have been intensively studied in recent years (for example, see [EG], [D], [EK1] - [EK3], [DGV], [DG99]). The basic mechanisms are *migration* between colonies in the diffusion limit this takes the form of deterministic mass flow between sites and then within each colony *resampling (i.e. pure genetic drift)* which gives the diffusion term, *selection* of haploid type based on a fitness function on the space of types and *mutation* between types occuring at rates given by a transition kernel on the space of types, the latter two in the diffusion limit are drift terms giving a deterministic massflow between types.

The purpose of this paper is to investigate how new types brought in by rare mutation invade and conquer the whole population through the composed effect of resampling, mutation, selection and migration.

The first part of the picture we focus on is a scenario (*punctuated equilibrium*) which has received considerable attention in the biological literature (see for example [EG1],[EG2] and the references below). In this scenario a population remains for long periods of time in a nearly stationary state (*quasi-equilibrium or stasis*) until the emergence of one or several rare mutants of higher fitness that then take over the population relatively quickly. The explanation of this phenomenon has two aspects tunnelling between local fitness maxima and the effect of the spatial structure.

For example one explanation of this tunnelling was given by Newman, Cohen and Kipnis [NCK] in the context of Wright's theory of evolution in an adaptive landscape (see [Wr1],[Wr2],[W]). In this explanation the long times between transitions from one quasi-equilibrium to another corresponds to "tunnelling" between one adaptive peak and another in a situation in which local stabilizing selection maintains a population near a local peak. In this setting, the Ventsel-Freidlin theory of small random perturbations of dynamical systems asserts that the actual transition, when it occurs, takes place rapidly. Thus this scenario would be consistent with the observations of paleontologists of periods of rapid evolution separated by very long periods of "stasis". This is a stochastic effect not present in the deterministic version, that is, the infinite population limit, but must be formulated in terms of a stochastic diffusion limit, which we consider here.

A second part of the picture is the role of geographic space and its strong effects on the qualitative behaviour. Namely typically a large population is subdivided into small subpopulations

(colonies) occupying different geographic regions. Then tunnelling between adaptive peaks can occur in these subpopulations and then spatial migration of individuals from a colony corresponding to a higher adaptive peak can spread and result in the take-over of the entire population which then fixates on the fitter type.

Our goal is to develop a rigorous framework to discuss a model incorporating these features of a punctuated equilibrium (by rare mutation) but with a different driving mechanism for the tunnelling starting from rare mutants than in the literature quoted above. In this context our goal is also more generally to give a framework which provides insight into the respective roles of mutation, selection and spatial migration in the emergence and spatial spread of rare mutants corresponding to a higher adaptive peak (quasi-equilibrium).

In this paper we develop the scenario in the *two-type* case, one type of low and one type of high fitness. The full picture meaning (1) an infinite hierarchy of fitness levels, (2) with $M \geq 2$ (instead of $M = 1$ here) types on the successive levels and (3) all this embedded on a countable geographic space instead of $\{1, 2, \cdots, N\}$, which approximates $\mathbb{Z}^2$ is developed in [DGsel].

Our goal in this paper is to exhibit how a rare mutant which is fitter than the types in the current population invades the population, by starting to take over a sparse set of colonies by selective advantage and then spreads as an increasing droplet in space till the complete population is finally taken over. The basic technique to do so is to consider the limit $N \to \infty$.

Indeed using this technique of taking $N \to \infty$ we can produce limiting objects, which have a simple and fairly explicit description. However due to the interaction between the migration, selection, mutation and resampling mechanisms, we have to develop some new ideas to adapt the method of multi-scale analysis to resolve some delicate conceptional and mathematical problems that arise in this case. (In previous work [DG99] the qualitative analysis of the longtime behaviour did not include mutation.)

A further point is that we give here a rigorously treated example for two-scale phenomena and develop as mathematical tool the notion of random nonlinear evolutions, more specifically solutions with *random McKean-Vlasov entrance laws* from time $-\infty$. The technique of random entrance laws allows to connect the separating time scales. This phenomenon is of relevance in many other applications.

In order to describe the early occurrence of rare mutants we work with a description of the sparse set of colonies conquered by the rare mutants via *atomic measure-valued processes* which in the limit $N \to \infty$ can be described by stochastic processes driven by (inhomogeneous) Poisson random measures and in order to calculate the intensity measure we are using excursion theory.

We have developed in [DGsel] a technique to extend this picture to infinite geographic space, multiple types on each level of fitness and infinite hierarchy of fitness levels. There we get a good set-up to study the scenario of punctuated equilibria. However this analysis still suffers some shortcomings. The restriction of our method is that we assume that the different time and space scales *separate* in the limit of large times and large space scales which is of course only an approximation but allows us to make precise the notion of a quasi-equilibrium as equilibrium of a limiting non-linear McKean-Vlasov dynamic. Moreover the techniques can be extended to the multitype case and then allow to define the concept of a *quasi-equilibrium* in a mathematically rigorous fashion as equilibria of certain limiting dynamics in the various time scales.

1.2 The model and key functionals

We begin here below with the description of the problem for the precisely stated model, define key functionals and formulate the problem.

1.2.1 The model

We define a system of N-exchangeably interacting finite populations ($d > 0$) with two types $\{1, 2\}$ with fitness 0 and 1 respectively, selection rate $s > 0$ and migration rate $c > 0$.

This means that we study the stochastic process $(X^N(t))_{t \geq 0}$, which is uniquely defined as follows:

$$X^N(t) = ((x_1^N(i,t), x_2^N(i,t)); i = 1, \ldots, N), \tag{1.1}$$

$$x_1^N(i,0) = 1, \quad i = 1, \cdots, N, \tag{1.2}$$

satisfying the wellknown SSDE where $i \in \{1, 2, \cdots, N\}$:

$$\begin{aligned} dx_1^N(i,t) = c(\bar{x}_1^N(t) - x_1^N(i,t))dt \quad &- \quad s\, x_1^N(i,t) x_2^N(i,t)dt - \frac{m}{N} x_1^N(i,t)dt \\ &+ \quad \sqrt{d \cdot x_1^N(i,t) x_2^N(i,t)}\, dw_1(i,t), \end{aligned} \tag{1.3}$$

$$\begin{aligned} dx_2^N(i,t) = c(\bar{x}_2^N(t) - x_2^N(i,t))dt \quad &+ \quad s\, x_1^N(i,t) x_2^N(i,t)dt + \frac{m}{N} x_1^N(i,t)dt \\ &+ \quad \sqrt{d \cdot x_1^N(i,t) x_2^N(i,t)}\, dw_2(i,t), \end{aligned} \tag{1.4}$$

where $\{(w_1(i,t))_{t \geq 0}, i = 1, \ldots, N\}$ are i.i.d. Brownian motions and w_2, w_1 are coupled via $w_2(i,t) = -w_1(i,t)$,

$$\widehat{x}_\ell(t) = \sum_{i=1}^N x_\ell^N(i,t) \text{ with } \ell = 1, 2 \tag{1.5}$$

and

$$\bar{x}_\ell(t) = \frac{1}{N} \widehat{x}_\ell(t) \text{ with } \ell = 1, 2. \tag{1.6}$$

Without loss of generality we can assume for the resampling rate d that $d = 1$ and will do this except where we wish to indicate the roles of the parameters c, s, d, m. The simple collection of N one-dimensional stochastic differential equations in (1.1)- (1.4) allows us to focus first on the key features of emergence, which are already complicated enough as we shall see.

Step 2 *(Key functionals and main objectives)*

The description of the system proceeds via four objects, namely either locally by a *sample of tagged sites* or globally by the *empirical measure* over the whole spatial collection and furthermore we use as well an *atomic measure* to globally describe sparse (in space) spots of type two mass respectively the *Palm measure* to describe this locally by formalizing the concept of a typical type-2 site. The asymptotic analysis of these four objects leads to propagation of chaos results, respectively, nonlinear McKean-Vlasov limiting dynamics and atomic measure-valued processes driven by Poisson processes and formulas for Palm measures based on this.

Since it often suffices to consider the tagged site 1, i.e. $x_\ell^N(1,t)$, in the sequel to designate a tagged site we set

$$x_\ell^N(t) = x_\ell^N(1,t) \quad ; \ell = 1, 2. \tag{1.7}$$

If we are interested in the *local* picture we consider for some $L \in \mathbb{N}$ the tagged sites

$$(x_\ell^N(1,t), \cdots, x_\ell^N(L,t)), \quad \ell = 1, 2. \tag{1.8}$$

We abbreviate the marginal law of the type-2 mass as

$$\mu_t^N := \mathcal{L}[x_2^N(1,t)] \in \mathcal{P}([0,1]). \tag{1.9}$$

The empirical measure process gives a *global* picture and is defined by:

$$\Xi_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{(x_1^N(i,t),x_2^N(i,t))} \in \mathcal{P}(\mathcal{P}(\{1,2\})) \tag{1.10}$$

and the empirical measure process of either type is defined by

$$\Xi_N(t,\ell) := \frac{1}{N}\sum_{i=1}^N \delta_{x_\ell^N(i,t)} \in \mathcal{P}([0,1]),\ \ell = 1,2. \tag{1.11}$$

Note that since $x_1^N(i,t) + x_2^N(i,t) = 1$ it suffices in the case of two types to know one component of the pair in (1.11), the other is then determined by this condition. Then we can for two types effectively replace $\mathcal{P}(\mathcal{P}(\{1,2\}))$ by $\mathcal{P}([0,1])$ as states of the empirical measure.

Next we have to describe the sparse sites where type 2 appears. First consider N sites starting only with type 1 mass at time 0 and evolving in the time interval $[0,T_0]$. As a result of the rare mutation from type 1 to type 2 at rate $\frac{m}{N}$ mutant mass will appear and then form non-negligible colonies at a sparse set of the total of N sites, but the typical site will have mass of order $o(1)$ as $N \to \infty$. We can now again take a *global* perspective, where we describe the sparse set of colonised (by type 2) sites or a *local* perspective, where we consider a typical colonised site via the concept of the Palm measure. We introduce both viewpoints successively.

Turn to the global perspective. In order to keep track of the sparse set of sites at which nontrivial mass appears we will give a random label to each site and define the following *atomic-measure-valued process.*

We assign independent of the process a point $a(j)$ randomly in $[0,1]$ to each site $j \in \{1,\dots,N\}$, that is, we define the collection

$$\{a(j), \quad j = 1,\cdots,N\} \text{ are i.i.d. uniform on } [0,1]. \tag{1.12}$$

We then associate with our process and a realization of the random labels a measure-valued process on $\mathcal{P}([0,1])$, which we denote by

$$\beth_t^{N,m} = \sum_{j=1}^N x_2^N(j,t)\delta_{a(j)}. \tag{1.13}$$

This description by $\beth_t^{N,m}$ is complemented by a local perspective, arising by zooming in on the random location where we actually find mass by studying a *typical* type-2 dominated site. In order to make precise the notion of a site as seen by a typical type-2 mass, i.e. a site seen from a randomly (among the total population in all N sites) chosen individual of type 2, we use the concept of the *Palm-distribution*, which we indicate by a hat. Define

$$\widehat{\mu}_t^N = \widehat{\mathcal{L}}[x_2^N(1,t)],\ \text{by}\ \widehat{\mu}_t^N(A) = \left(\int_A x\mu_t^N(dx)\right) / E[x_2^N(1,t)], \tag{1.14}$$

where $\mu_t^N = \mathcal{L}[x_2^N(1,t)]$. The law $\widehat{\mu}_t^N$ describes now for an i.i.d. initial state the law of a *site, typical* for the type-2 population at time t in the limit $N \to \infty$. Namely we can think of this as picking a type-2 individual at random and then determine the site where it sits. Note that

$$\int x\widehat{\mu}_t^N(dx) = \frac{E[(x_2^N(1,t))^2]}{E[x_2^N(1,t)]}. \tag{1.15}$$

The scenario

The basic objective of this paper is to describe the emergence of the more fit type, i.e. type 2, using the above functionals of the process. This emergence occurs in *three regimes*, two are of short duration separated by a third of long duration and arise as follows in an asymptotic description as $N \to \infty$.

(1) In the first regime during times of order $O(1)$ rare mutants arise at a small set of sites (droplet) where they locally make up substantial part of the population and then develop mutant "droplets" by migration. We will see later in Subsection 3.3 in all detail that the mutant population during this stage develops as a type of measure-valued branching process. We refer to this stage as *droplet formation*.

(2) The next stage, namely, the stage at which type 2 has very small but $O(1)$ frequency at a typical (i.e. randomly chosen site) site, or equivalently at the macroscopic level has $O(1)$ density, occurs at time $O(\log N)$, we refer to this as *emergence*.

(3) The last regime at which the macroscopic density of type 2 reaches a frequency arbitrarily close to 1 is taking a further piece of time but is again only of order $O(1)$ and is referred to as *fixation*. For the regime between emergence and fixation we give a description by a limiting evolution in the form of an entrance law evolving from time $-\infty$ and type-1 states to type-2 states at time $+\infty$ and hence altogether with time index $t \in \mathbb{R}$.

In the sequel we will analyse these three regimes of droplet formation, emergence and fixation by studying the behaviour as $N \to \infty$ of

$$(1.16) \qquad \Xi_N(t), \quad \{x_\ell^N(i,t); \ell = 1,2; i = 1,\cdots,L\} \text{ and } \daleth_t^{N,m}, \quad \widehat{\mu}_t^N,$$

in finite times as well as in larger N-dependent time scales thus capturing each of the three regimes described above.

1.3 Statement of results

We have three tasks, first to determine the *macroscopic emergence time scale*, that is the time scale at which the level two type appears at a typical site or equivalently has positive intensity, second, to describe the pre-emergence picture, in particular the *droplet formation* respectively the *limiting droplet dynamics* and thirdly to describe the *limiting dynamic of fixation* after the emergence with which the take-over by type 2 then occurs.

We begin the analysis with introducing the two different limit dynamics in the subsequent two subsubsections and then subsequently the emergence, fixation and convergence results are stated in a further subsubsection.

1.3.1 The limiting random McKean-Vlasov dynamics and random entrance laws

We start with the initial condition $\Xi_N(0,\{1\}) = \delta_1$ (recall (1.11)), that is only type 1 appears. The objective is to establish the emergence and fixation of type 2 in times of the form $t_N = C \cdot \log N + t$, to identify the constant C, and to identify the limit of $\Xi_N(t_N,\cdot))$ as a function of t in the process of emergence and determine the dynamics in t of the fixation (takeover) which leads to the concentration of the complete population in type-2 as $N \to \infty$. In the next subsubsection we shall then discuss the behaviour at early times, i.e. times t_N with $\limsup((\log t_N)/\log N) < C$, in the stage of droplet formation.

The key ingredients in the limiting processes in times t_N for the local and global description are:

- the limiting *(nonlinear) McKean-Vlasov dynamics*,
- the *entrance law* from $-\infty$ of the McKean-Vlasov dynamic and

- *random* solutions to the McKean-Vlasov equation.

In order to define these ingredients we proceed in three steps, we recall first in Step 1 the "classical" McKean-Vlasov limit (associated with a nonlinear Markov process) before in Step 2 and Step 3 we introduce the two new objects.

Step 1 Consider the above system (1.1)-(1.4) of N interacting sites with type space $\mathbb{K} = \{1, 2\}$ starting at time $t = 0$ from a product measure (that is, i.i.d. initial values at the N sites). The basic McKean-Vlasov limit (cf. [DG99], Theorem 9) says that if we start initially in an i.i.d. distribution, then

$$(1.17) \qquad \{\Xi_N(t)\}_{0 \le t \le T} \underset{N \to \infty}{\Longrightarrow} \{\mathcal{L}_t\}_{0 \le t \le T},$$

where the $\mathcal{P}(\mathcal{P}(\mathbb{K}))$-valued path $\{\mathcal{L}_t\}_{0 \le t \le T}$ is the law of a *nonlinear* Markov process, namely the unique weak solution of the *McKean-Vlasov equation*:

$$(1.18) \qquad \frac{d\mathcal{L}_t}{dt} = (L_t^{\mathcal{L}_t})^* \mathcal{L}_t,$$

where for $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{K}))$, and $F(\mu) = f(\mu(2))$,

$$(1.19) \qquad L^\pi F = c[\int y\pi(2, dy) - x]\frac{\partial f}{\partial x} + sx(1-x)\frac{\partial f}{\partial x} + \frac{d}{2}x(1-x)\frac{\partial^2 f}{\partial x^2}$$

and the $*$ indicates the adjoint of an operator mapping from a dense subspace of $C_b(E, \mathbb{R})$ into $C_b(E, \mathbb{R})$ w.r.t. the pairing of $\mathcal{P}(E)$ and $C_b(E, \mathbb{R})$ given by the integral of the function with respect to the measure. Similar equations have been studied extensively in the literature (e.g. [Gar]). The process $(\mathcal{L}_t)_{t \ge 0}$ corresponds to a nonlinear Markov process since $\mathcal{L}_t$ appears also in the expression for the generator L.

As pointed out above in (1.11), in the special case $\mathbb{K} = \{1, 2\}$, we can simplify by considering the frequency of type 2 only and by reformulating (1.18) living on $\mathcal{P}(\mathcal{P}(\{1, 2, \}))$ in terms of $\mathcal{L}_t(2) \in \mathcal{P}[0, 1]$. This we carry out now.

Namely we note that given the mean-curve

$$(1.20) \qquad m(t) = \int\limits_{[0,1]} y\, \mathcal{L}_t(2)(dy),$$

the process $(\mathcal{L}_t(2))_{t \ge 0}$ is the *law* of the strong solution of (i.e. the unique weak solution) the SDE:

$$(1.21) \qquad dy(t) = c(m(t) - y(t))dt + sy(t)(1 - y(t))dt + \sqrt{dy(t)(1 - y(t)}dw(t).$$

Then informally $(\mathcal{L}_t)_{t \ge 0}$ corresponds to the solution of the nonlinear diffusion equation. Namely for $t > 0$, $\mathcal{L}_t(2)(\cdot)$ is absolutely continuous and for

$$(1.22) \qquad \mathcal{L}_t(2)(dx) = u(t, x)dx \ \in \mathcal{P}([0, 1])$$

the evolution equation of the density $u(t, \cdot)$ is given by:

$$(1.23) \qquad \frac{\partial}{\partial t}u(t, x) = -c\frac{\partial}{\partial x}\{[\int\limits_{[0,1]} yu(t, y)dy - x]u(t, x)\} - s\frac{\partial}{\partial x}(x(1-x)u(t, x)) + \frac{d}{2}\frac{\partial^2}{\partial x^2}(x(1-x)u(t, x)).$$

Step 2 In order to describe the emergence and invasion process via Ξ_N, we introduce in this step and the next two extensions of the nonlinear McKean-Vlasov dynamics which describes only the limiting evolution over finite time stretches, $t \in [t_0, t_0 + T]$ given an initial condition $\mathcal{L}_{t_0}$ which

we then follow as $t_0 \to -\infty$. Hence we have to define the dynamics for $t \in (-\infty, \infty)$ in terms of an *entrance law* at $-\infty$.

Since we consider the limits of systems observed in the interval $const \cdot \log N + [-\frac{T}{2}, \frac{T}{2}]$ with T any positive number, that is setting $t_0(N) = const \cdot \log N - \frac{T}{2}$, we need to identify entrance laws for the process from $-\infty$ (by considering $T \to \infty$) out of the state concentrated on type 1 with certain properties.

Definition 1.1 *(Entrance law from $t = -\infty$)*

We say in the two-type case that a probability measure-valued function $\mathcal{L} : \mathbb{R} \to \mathcal{P}([0,1])$, is an entrance law at $-\infty$ starting from type 1 if $(\mathcal{L}_t)_{t \in (-\infty,\infty)}$ is such that $\mathcal{L}_t$ solves the McKean-Vlasov equation (1.18) and $\mathcal{L}_t \to \delta_1$ as $t \to -\infty$.

In the case of more than two types we work with maps $\mathcal{L} : \mathbb{R} \to \mathcal{P}(\mathcal{P}(\mathbb{K}))$ where we require as $t \to -\infty$ that $\mathcal{L}_t$ converges to measures δ_μ with $\mu \in \mathcal{P}(\mathbb{K})$ such that μ is a measure concentrated on the lower level types. $\square$

The existence of such an object is obtained in part (c) of the proposition below.

Step 3 The usual formulation of the McKean-Vlasov limit requires that we start in an i.i.d. initial configuration. This is not sufficient for us since even though we consider systems observed in finite time stretches we do so only *after large times*, namely, after the time $const \cdot \log N$ with $N \to \infty$. Hence in our context the McKean-Vlasov limit is valid in a fixed time scale but if the system is viewed in time scales that depend on N this can (and will) break down since we only know that the initial state is exchangeable and this then leads to a random solution.

We will indeed establish that the emergence of rare mutants gives rise to *"random"* solutions of the McKean-Vlasov dynamics. In particular we will show that the limiting empirical measures at times of the form $C \log N + t$ are *random* probability measures on $[0,1]$ and therefore given by exchangeable sequences of $[0,1]$- valued truely exchangeable random variables which are *not* i.i.d., that is, the exchangeable σ-algebra is not trivial. This means that the empirical mean turns out to be a *random variable* and since this is the term driving via the migration the local evolution of a site in the McKean-Vlasov limit, the non-linearity of the evolution equation comes seriously into play. However once we condition on the exchangeable σ-algebra, we then get for the further evolution again a deterministic limiting equation for the empirical measures, namely the McKean-Vlasov equation. The reason for this is the fact that conditioned on the exchangeable σ-algebra we obtain on asymptotically (as $N \to \infty$) i.i.d. configuration to which the classical convergence theorem applies. Using the Feller property of the system, which is a direct consequence of the duality, we get our claim.

This leads to the task of identifying an entrance law in terms of a *random initial condition at time* $-\infty$. A consequence of this scenario is that when we use the duality from time T_N to $T_N + t$, we apply it to a random initial state in the limit and we therefore have to use because of the non-linearity of the evolution the appropriate formulas.

The above discussion shows that we need to introduce the notion of a truly random McKean-Vlasov entrance law from $-\infty$.

Definition 1.2 *(Random solution of McKean-Vlasov)*

We say that the probability measure-valued process $\{\mathcal{L}(t)\}_{t \in \mathbb{R}}$ is a random solution of the McKean-Vlasov equation *(1.18) if*

- *$\{\mathcal{L}_t : t \in \mathbb{R}\}$ is a.s. a solution to (1.18), that is, for every t_0 the distribution of $\{\mathcal{L}_t : t \geq t_0\}$ conditioned on $\mathcal{F}_{t_0} = \sigma\{\mathcal{L}_s : s \leq t_0\}$ is given by $\delta_{\{\mu_t\}_{t \geq t_0}}$ where μ_t is a solution of the McKean-Vlasov equation with $\mu_{t_0} = \mathcal{L}_{t_0}$,*
- *the time t marginal distributions of $\{\mathcal{L}_t : t \in \mathbb{R}\}$ are truly random.* $\square$

The key result of this subsubsection on the objects introduced in the previous three steps is now the following existence and uniqueness results on the solution of the McKean-Vlasov equation (1.18) and of its (random) entrance laws from $t = -\infty$:

Proposition 1.3 *(McKean-Vlasov entrance law from $-\infty$)*
(a) Given the initial state $\mu_0 \in \mathcal{P}([0,1])$ there exists a unique solution

$$\mathcal{L}_t(2)(dx) = \mu_t(dx), \quad t \geq t_0 \tag{1.24}$$

to (1.18) with initial condition $\mathcal{L}_{t_0}(2) = \mu_0$.
(b) If $s > 0$ and $\int_{[0,1]} x\mu_{t_0}(dx) > 0$, then this solution satisfies:

$$\lim_{t\to\infty} \mathcal{L}_t(2)(dx) = \delta_1(dx). \tag{1.25}$$

(c) There exists a solution $(\mathcal{L}_t^{**}(2))_{t\in\mathbb{R}}$ to equation (1.18) satisfying the conditions:

$$\begin{aligned} \lim_{t\to-\infty} \mathcal{L}_t^{**}(2) &= \delta_0, \\ \lim_{t\to\infty} \mathcal{L}_t^{**}(2) &= \delta_1 \\ \int_{[0,1]} x\mathcal{L}_0^{**}(2,dx) &= \frac{1}{2}. \end{aligned} \tag{1.26}$$

This solution is called an entrance law from $-\infty$ with mean $\frac{1}{2}$ at $t = 0$.
(d) We can obtain a solution in (c) such that:

$$\exists\, \alpha \in (0,s) \text{ and } A_0 \in (0,\infty) \text{ such that } \lim_{t\to-\infty} e^{\alpha|t|} \int_{[0,1]} x\mathcal{L}_t^{**}(2,dx) = A_0. \tag{1.27}$$

(e) The solution of (1.18) also satisfying (1.27) for prescribed A_0 is unique and if $A_0 \in (0,\infty)$ then α is necessarily uniquely determined.

For any deterministic solution

$$\mathcal{L}_t, \ t \in \mathbb{R} \tag{1.28}$$

to (1.18) with

$$0 \leq \limsup_{t\to-\infty} e^{\alpha|t|} \int_{[0,1]} x\mathcal{L}_t(2,dx) < \infty, \tag{1.29}$$

the limit $A = \lim_{t\to-\infty} e^{\alpha|t|} \int_{[0,1]} x\mathcal{L}_t(2,dx)$ exists.

If $A > 0$, then $\{\mathcal{L}_t, t \in \mathbb{R}\}$ is given by a time shift of the then unique $\{\mathcal{L}_t^{**}, t \in \mathbb{R}\}$ singled out in (1.27), i.e.

$$\mathcal{L}_t = \mathcal{L}_{t+\tau}^{**}, \quad \tau = \alpha^{-1} \log \frac{A}{A_0}. \tag{1.30}$$

For future reference we define $(\mathcal{L}_t^*)_{t\in\mathbb{R}}$ to be the unique solution satisfying

$$\lim_{t\to-\infty} e^{\alpha|t|} \int_{[0,1]} x\mathcal{L}_t^*(2,dx) = 1 \text{ for some } \alpha \in (0,s). \tag{1.31}$$

(f) Any random solution $(\widetilde{\mathcal{L}}_t)_{t\in\mathbb{R}}$ to (1.18) such that

$$\limsup_{t\to-\infty} e^{\alpha|t|} E[\int_{[0,1]} x\mathcal{L}_t(2,dx)] < \infty, \quad \liminf_{t\to-\infty} e^{\alpha|t|} [\int_{[0,1]} x\mathcal{L}_t(2,dx)] > 0 \text{ a.s.}, \tag{1.32}$$

is a random time shift of $(\mathcal{L}_t^{**})_{t\in\mathbb{R}}$ (and of $(\mathcal{L}_t^*)_{t\in\mathbb{R}}$). $\square$

Example 1 *Let $\mathcal{L}_t^*$ be a solution satisfying (1.23), (1.27) and for a given value of A let τ be a true real-valued random variable. Then $\{\mathcal{L}_{t-\tau}^*\}_{t\in\mathbb{R}}$ is a truly random solution. This can also be viewed as saying that we have a solution with an exponential growth factor A which is truly random.*

Remark 1 *We shall derive later on a bound from above on the cardinality of our dual process which will imply that the first relation in (1.32) must be satisfied for a limiting dynamic (as $N \to \infty$) of our empirical measures Ξ_N and furthermore we get a lower bound implying that A is a.s. positive for this limiting dynamic, (see (3.146) and Subsubsection 3.2.6). Then we are able to use the identification of the solution given in (1.30) or of the one satisfying (1.31) to identify the limiting dynamic of the process of empirical measures.*

1.3.2 Limiting dynamics of sparse sites with substantial type-2 mass: Droplets

We next describe the limiting dynamic of the sparse set of sites which have been colonized by type 2 prior to the onset of emergence. There are two time regimes, first an initial finite time horizon of order $O(1)$, then large times $t_N \to \infty$ as $N \to \infty$ but which remain $<< \alpha^{-1}\log N$ so that we are in the preemergence regime with still a global density of type 2 which is asymptotically zero.

It is very unlikely that a randomly chosen site has at time O(1) mass of type 2 at least $\varepsilon > 0$ as $N \to \infty$. In fact an explicit calculation (see (3.703)) shows that the transition density decays like $constN^{-1}$. On the other hand the number of sites increases with N so that here we have a compensation provided we do not look at only one single tagged site but the complete population at all sites and the result is that considering all sites there is a *finite random number* of sites with substantial type 2 mass. Newly added sites in this set of sites arise from a process entering from the state 0. In fact we obtain here a Poisson distribution in the limit. This whole scenario will be made precise using entrance laws from state 0 which we derive using diffusion theory and excursion law theory.

Therefore before formulating the main results of this section we first consider entrance laws of a single site with no mutation from type 1 to type 2 which provides us with a key ingredient, the *excursion measure* $\mathbb{Q}$. We state the wellknown fact:

Lemma 1.4 *(Single site: entrance and excursion laws)*

(a) Let $c > 0, d > 0,\ s > 0$. Then 0 is an exit boundary for the the Fisher-Wright diffusion

$$dx(t) = -cx(t)dt + sx(t)(1-x(t))dt + \sqrt{d\cdot x(t)(1-x(t))}dw(t), \tag{1.33}$$

which then has a σ-finite entrance law from state 0 at time 0, the σ-finite excursion law

$$\mathbb{Q} = \mathbb{Q}^{c,d,s} \tag{1.34}$$

on

$$W_0 := \{w \in C([0,\infty),\mathbb{R}^+),\ w(0) = 0,\ w(t) > 0 \text{ for } 0 < t < \zeta \text{ for some } \zeta \in (0,\infty)\}. \tag{1.35}$$

(b) Moreover, denoting by P^ε the law of the process started with $w(0) = \varepsilon$ and $\varepsilon > 0, \mathbb{Q}$ is given by:

$$\mathbb{Q}(\cdot) = \lim_{\varepsilon\to 0}\frac{P^\varepsilon(\cdot)}{S(\varepsilon)}, \tag{1.36}$$

where $S(\cdot)$ is the scale function of the diffusion (1.33), defined by the relation,

$$P_\varepsilon(T_\eta < \infty) = \frac{S(\varepsilon)}{S(\eta)}, \qquad 0 < \varepsilon < \eta < \infty, \tag{1.37}$$

where T_η is the first hitting time of η.

For the Fisher-Wright diffusion S is given by (cf. [RW], V28)) the initial value problem:

$$(1.38) \qquad S(0) = 0, \frac{dS}{dx} = \frac{e^{-2sx}}{(1-x)^{2c}},$$

so that

$$(1.39) \qquad \lim_{\varepsilon \to 0} \frac{S(\varepsilon)}{\varepsilon} = 1.$$

(c) The measure $\mathbb{Q}$ is σ-finite, namely for any $\eta > 0, \zeta$ as in (1.35),

$$(1.40) \qquad \mathbb{Q}(\{w : \zeta(w) > \eta\}) < \infty,$$

$$(1.41) \qquad \mathbb{Q}(\sup_t(w(t)) > \eta) = \mathbb{Q}(T_\eta < \infty) = \frac{1}{S(\eta)} \longrightarrow \infty \text{ as } \eta \to 0,$$

and

$$(1.42) \qquad \int_0^1 x \mathbb{Q}(\sup_t(w(t)) \in dx) = \infty. \qquad \square$$

Proof It is wellknown that 0 is an exit boundary for (1.33). In this case (a) - (c) then follows immediately from the results of Pitman and Yor [PY], Section 3.

We next identify the limit in distribution of $\daleth_t^{N,m}$ (defined in (1.13)) as $N \to \infty$. Let

$$(1.43) \qquad \mathcal{M}_a([0,1]) = \text{ the set of finite atomic measures on } [0,1],$$

equipped with the weak atomic topology (see [EK4] and in this present work in (1.57)-(1.63) below for details on this topology) which forms a Polish space.

We will need Poisson random measures in four variables, the actual time called $s \in [0,t]$, the location parameter $a \in [0,1]$, the mutation and immigration potential called $u \in [0,\infty)$ and the path of an excursion called $w \in W_0$. This allows us to obtain the droplet dynamic $(\daleth_t^m)_{t \in [0,\infty)}$ and its properties:

Proposition 1.5 *(A continuous atomic-measure-valued Markov process)*

Let $\daleth_0^*(t) = \sum_i y_i(t)\delta_{a_i} \in \mathcal{M}_a([0,1])$ where the $y_i(t)$ are independent solutions of the SDE (1.33) (describing the initial droplet and its evolution) and let $N(ds, da, du, dw)$ be a Poisson random measure on (recall (1.35) for W_0)

$$(1.44) \qquad [0,\infty) \times [0,1] \times [0,\infty) \times W_0,$$

with intensity measure

$$(1.45) \qquad ds\, da\, du\, \mathbb{Q}(dw),$$

where $\mathbb{Q}$ is the single site excursion law defined in (1.36) in Lemma 1.4.

Then the following three properties hold.

(a) The stochastic integral equation for $(\daleth_t^m)_{t \geq 0}$ a process with values in $\mathcal{M}_a([0,1])$ is given as

$$(1.46) \qquad \daleth_t^m = \daleth_0^*(t) + \int_0^t \int_{[0,1]} \int_0^{q(s,a)} \int_{W_0} w(t-s)\delta_a N(ds, da, du, dw), \quad t \geq 0,$$

where $\delta_a(\cdot) \in \mathcal{M}_1([0,1])$ and where $q(s,a)$ denotes the non-negative predictable function

$$q(s,a) := (m + c\gimel^m_{s-}([0,1])), \tag{1.47}$$

has a unique continuous $\mathcal{M}_a([0,1])$-valued solution, which we call

$$(\gimel^m_t)_{t \geq 0}. \tag{1.48}$$

(b) $(\gimel^m_t)_{t\geq 0}$ is a $\mathcal{M}_a([0,1])$-valued strong Markov process.
(c) The process $(\gimel^m_t)_{t\geq 0}$ has the following properties:

- *the mass of each atom observed from the time of its creation follows an excursion from zero generated from the excursion law $\mathbb{Q}$ (see (1.36)),*
- *new excursions are produced at time t at rate*

$$m + c\gimel^m_t([0,1]), \tag{1.49}$$

- *each new excursion produces an atom located at a point $a \in [0,1]$ chosen according to the uniform distribution on $[0,1]$,*
- *at each t for $\varepsilon > 0$ there are at most finitely many atoms of size $\geq \varepsilon$,*
- *$t \to \gimel^m_t([0,1])$ is a.s. continuous.* □

Remark 2 *The process $(\gimel^m_t)_{t\geq 0}$ can be viewed as a continuous state analogue of the Crump-Mode-Jagers branching process with immigration (see Subsubsection 3.2.4, Step 2 for a review and more information on this type of processes). We shall see in (1.79) that the total mass grows exponentially as in a supercritical branching process.* □

Remark 3 *A necessary and sufficient condition for extinction of the analogue of $\gimel^*_t$ for a general class of one-dimensional diffusions was obtained by M. Hutzenthaler ([Hu]). In the Fisher-Wright case with $c > 0,\ s > 0$ the fact that the probability of non-extinction is non-zero follows from the Proposition 1.11.* □

Proof (a) and (b) The existence and uniqueness of the solution to (1.46) and the strong Markov property follows as in [D-Li] and [FL].

(c) follows directly from the construction via the Poisson measure N given in (1.46) .

Remark 4 *We can enrich the process $(\gimel^m_t)_{t\geq 0}$ to include the genealogical information, namely which mass results from which mutation. For that purpose we have in addition to the location record the birth times of atoms due to migration (successful colonization). This means we have to split the rate at which excursions are created into mutation at the site and immigration from other sites. At time 0 we start with*

$$\widehat{\gimel}^*_0(t) \tag{1.50}$$

describing the initial atoms we consider to be present at time 0.

The genealogical enrichment denoted

$$(\widehat{\gimel}^m_t)_{t\geq 0} \tag{1.51}$$

is a measure-valued process on a richer set E and is obtained as follows.

Let

$$E = ((\mathbb{R}_+ \times [0,1]) \cup \emptyset)^{\mathbb{N}} \tag{1.52}$$

and for $y \in E$ define $\tau(y) = min\{n : y(n) = \emptyset\} - 1$.

Let $N(ds, da, du, dw)$ be a Poisson random measure on $[0,\infty)\times[0,1]\times[0,\infty)\times W_0$ with intensity measure $ds\, da\, du\, \mathbb{Q}(dw)$ where $\mathbb{Q}$ is the single site excursion law. Then the following stochastic integral equation has a unique continuous solution, $\widehat{\daleth}_t^m$:

$$\widehat{\daleth}_t^m = \widehat{\daleth}_0^*(t) + \int_0^t \int_E \int_{[0,1]} \int_0^{q(s,y,a)} \int_{W_0} w(t-s)\delta_{y\diamond(s,a)} N(ds, da, du, dw), \tag{1.53}$$

where (s,a) is short for $((s,a), \emptyset, \emptyset, \cdots)$,

$$q(s,y,a) = m \text{ if } \tau(y) = 0, \quad q(s,y,a) := c\widehat{\daleth}_{s-}^m(y),\ \tau(y) \neq 0, \tag{1.54}$$

and

$$\begin{array}{l} \diamond : E\times E \to E,\ y_1 \diamond y_2 = (y_1', y_2', \underline{\emptyset}) \quad (\textit{concatenation}) \textit{ with } , \\ y' = (y'(1), \cdots, y'(\tau(y'))) \quad , \underline{\emptyset} = (\emptyset, \emptyset, \cdots). \end{array} \tag{1.55}$$

Here $\widehat{\daleth}_t^m$ is a measure-valued process on E and $y \diamond (s,a)$ denotes the offspring of y with birth time s and location $a \in [0,1]$. Moreover, $\widehat{\daleth}_t^m(\{y : \tau(y) = \infty\}) = 0$ a.s..

Some topological facts concerning atomic measures. We finally recall the definition and some facts on the *weak atomic topology with metric ρ_a* on the space of finite atomic measures $\mathcal{M}_a([0,1])$ due to Ethier and Kurtz. Recall that we have the topology on $\mathcal{M}([0,1])$ induced by the weak topology which is induced by the Prohorov metric ρ. Next choose a function Ψ, where $\Psi : [0,\infty) \to [0,1]$ is continuous, nonincreasing and $\Psi(0) = 1,\ \Psi(1) = 0$. Then for $\nu, \mu \in \mathcal{M}_a([0,1])$ one defines

$$\begin{aligned} \rho_a(\nu,\mu) &:= \rho(\nu,\mu) \\ &+ \sup_{0<\varepsilon\leq 1} \left| \int_{[0,1]}\int_{[0,1]} \Psi(\frac{|x-y|}{\varepsilon} \wedge 1)\mu(dx)\mu(dy) - \int_{[0,1]}\int_{[0,1]} \Psi(\frac{|x-y|}{\varepsilon}\wedge 1)\nu(dx)\nu(dy) \right|. \end{aligned} \tag{1.56}$$

We refer to ρ_a as the *Ethier-Kurtz metric.*

The space $(\mathcal{M}_a([0,1]), \rho_a)$ is a Polish space and the topology, in other words convergence, does not depend on the choice of Ψ (the geometry of the space of course does).

The following lemma collects what we need on the relation between the weak and the weak atomic topologies, which are different.

Lemma 1.6 *(Weak atomic topology and weak topology)*

(a) A sequence of random finite atomic measures on $[0,1]$, $(\mu_n)_{n\in\mathbb{N}}$, converges to μ in the weak atomic topology if and only if

$$\mu_n \text{ converges weakly to } \mu \tag{1.57}$$

and

$$\mu_n^*([0,1]) \underset{n\to\infty}{\longrightarrow} \mu^*([0,1]), \text{ where } \quad \mu^*([0,1]) = \sum_{x\in[0,1]} \mu(\{x\})^2\delta_x. \tag{1.58}$$

(b) If the following three properties hold for $\mu_n, \mu \in \mathcal{M}_a([0,1])$ (here $\Longrightarrow$ denotes weak convergence as $n \to \infty$):

$$\mu_n \Rightarrow \mu, \text{ the ordered atom sizes converge and the set of atom locations converges,} \tag{1.59}$$

then:

$$\rho_a(\mu_n, \mu) \to 0. \tag{1.60}$$

(c) A continuous $\mathcal{M}_f$-valued process with a.s. continuous (in the weak topology) sample paths of the form $\sum_i a_i(t)\delta_{x_i(t)}$ such that $\sum a_i^2(t)$ is a.s. continuous, has also sample paths a.s. in $C([0,\infty),(\mathcal{M}_f([0,1],\rho_a))$, where ρ_a is the Ethier-Kurtz metric.
(d) Consider a sequence $\{Z_N, N \in \mathbb{N}\}$ of atomic measure-valued processes with cadlag paths in the weak atomic topology, so that Z_N has the form

$$Z_N(t) = \sum_i a_{N,i}(t)\delta_{x_{N,i}(t)} \tag{1.61}$$

for suitable functions (of t) $a_{N,i}, x_{N,i}$. Assume furthermore that

$$\{\mathcal{L}(Z_N), N \in \mathbb{N}\} \text{ in } \mathcal{P}(D([0,\infty),(\mathcal{M}_f([0,1],\rho))) \text{ is relatively compact.} \tag{1.62}$$

Then the compact containment condition will hold also in $(\mathcal{M}_a([0,1]),\rho_a)$, if and only if for each $T>0$ and $\delta>0$ there exists $\varepsilon>0$ such that

$$\inf_N P\left[\sup_{t\le T}\left(\int_{[0,1]}\int_{[0,1]}\Psi(\frac{|x-y|}{\varepsilon}\wedge 1)Z_N(t,dx)Z_N(t,dy) - \sum_i a_{N,i}^2(t)\right) \le \delta\right] \ge 1-\delta. \tag{1.63}$$

Here Ψ is as in (1.56).

Proof This follows from work by Ethier and Kurtz [EK4], namely (a) - Lemma 2.2, (b) -Lemma 2.5, (c) - Lemma 2.11, (d) - Remark 2.13.

1.3.3 Emergence time, droplet formation, fixation dynamic for N-interacting sites: Statement of results

We now have the ingredients and the background to continue the analysis and state all the results of the finite population model with two types $\mathbb{K} = \{1,2\}$ with fitness 0 and 1, respectively, $s>0$ and $d>0$ and with N exchangeable sites and with $c>0$. We start with only type 1 present initially. The exposition has three parts, the *emergence*, the *preemergence* (droplet formation) and the *fixation*.

Part 1: Emergence

The goal is to describe in mathematical precise form the initial formation of germs for the expression of the fitter type 2 by mutation which subsequently expand which then leads finally to the global emergence of the fitter type, and over the period of order $O(1)$ the increase of the mass of this type continues until it takes over almost the entire population (fixation). We state the results on this scenario in four main propositions.

First on emergence are Proposition 1.7 which shows that (global) emergence occurs at times of order $const \cdot \log N$ and which identifies the constant and then Proposition 1.12 and Proposition 1.15 which identify the limiting dynamics in t after global emergence of type 2, meaning we study the system observed in times $const \cdot \log N + t$ with $t \in \mathbb{R}$. The Proposition 1.9 explains the emergence behaviour by describing the very early formation of droplets of "type-2 colonies".

In addition to the four main statements mentioned above the exact properties in the early stage of droplet formation are given in Proposition 1.10 and Proposition 1.11. Namely droplet formation occurs in a random manner in the very beginning followed by a deterministic expansion of the droplet size leading to emergence on a global level.

A remarkable fact we state after the part on fixation in Proposition 1.14, which is establishing a close connection between exit behaviour in short scale and entrance behaviour in the same scale but placed much later such that both time intervals are separated by a long time stretch.

We begin with the emergence times.

Proposition 1.7 *(Macroscopic emergence and fixation times)*
(a) *(Emergence-time)*
There exists a constant α with:

(1.64) $$0 < \alpha < s,$$

such that if $T_N = \frac{1}{\alpha} \log N$, then for $t \in \mathbb{R}$ and asymptotically as $N \to \infty$ type 2 is present at times $T_N + t$, i.e. there exists an $\varepsilon > 0$ such that

(1.65) $$\liminf_{N\to\infty} P[x_2^N(T_N + t) > \varepsilon] > 0,$$

and type 2 is not present earlier, namely for $1 > \varepsilon > 0$:

(1.66) $$\lim_{t\to-\infty} \limsup_{N\to\infty} [P(x_1^N(T_N + t) < 1 - \varepsilon] = 0.$$

(b) *(Fixation time)*
After emergence the fixation occurs in times $O(1)$ as $N \to \infty$, i.e. for any $\varepsilon > 0$

(1.67) $$\lim_{t\to\infty} \limsup_{N\to\infty} P[x_1^N(T_N + t) > \varepsilon] = 0.$$

(c) *The constant α can be characterized as the Malthusian parameter for a Crump-Mode-Jagers branching process denoted $(\widetilde{K}_t)_{t\geq 0}$ which is explicitly defined below in (3.139). The constant α can alternatively be introduced as exponential growth rate for the limiting droplet growth dynamic $\beth_t^m$, see (1.80) or in terms of the excursion measure of a diffusion as specified below in (3.684).* □

Corollary 1.8 *(Emergence and fixation times of spatial density)*
The relations (1.65), (1.66) and (1.67) hold for $\bar{x}_2^N$ respectively $\bar{x}_1^N$ as well. □

Part 2: Droplet formation

This raises the question how the global emergence of type 2 actually came about and what is the role of α in the forward dynamics (instead of the view back from emergence) and why do we have a random element in the emergence. We will now demonstrate that rare mutation and subsequent selection with the help of migration produce in times $O(1)$ a cloud of sites where the type-2 is already manifest with substantial mass at a given time and this cloud then starts growing as time increases. This growth of total type-2 mass is exponential with a random factor arising as usual in the very beginning (i.e. at times growing arbitrarily slow with N) and hence in particular the growth process is given by a randomly shifted exponential. We call the growing cloud of type-2 sites a droplet (which in fact in the euclidian geographic space is literally accurate). The droplet will be described using the atomic measure $\beth_t^{N,m}$ introduced in (1.13) and by the Palm measure explained in (1.14).

For the purpose of making our three regime scenario precise we first need to investigate how for a finite time horizon the site of a typical type-2 mass looks like in the limit $N \to \infty$ and if we can show that such a site exhibits nontrivial type-2 mass. Then we have to see how fast the number of such sites grows and reaches size εN at time $\alpha^{-1} \log N + t(\varepsilon)$.

One might expect that a growth of the droplet at exponential rate α would be the appropriate scenario. This scenario will also show that some randomness created initially, i.e. times $O(1)$ as

$N \to \infty$, remains in the system up to fixation of type 2. This randomness arises since up to some finite time T there will be among the N-sites as $N \to \infty, O(1)$ such sites where the mass of type 2 exceeds some level $\delta > 0$. In the limit this will result in a compound Poisson number of germs for expansion. Only descendants of this early mass will make up at much later times the bulk of the type-2 mass, since we have exponential growth.

Therefore we study

- the configuration of type-2 mass on the sparse set of sites and its limiting law as $N \to \infty$ for a finite time horizon, i.e. $\beth_t^{N,m}$ and $\widehat{\mu}_t^N$ for $t \in [0,T]$,
- the configuration in a typical type-2 site as time and N tends slowly to infinity, i.e. $\widehat{\mu}_{t_N}^N$, with $t_N \uparrow \infty$ but $t_N = o(\log N)$,
- the growth of the droplet as time goes to infinity but slower than the time needed for global emergence, i.e. $\beth_{t_N}^{N,m}$ and $\beth_{t_N}^{N,m}([0,1])$, with t_N as above.

To verify our scenario we want to show that $\widehat{\mu}_{t_N}^N$ converges for $t_N \uparrow \infty$ to a limit, say $\widehat{\mu}_\infty^\infty$ which is nontrivial, i.e. $\widehat{\mu}_\infty^\infty((0,1)) = 1$. Then we want to prove that the number of sites looking like $\widehat{\mu}_\infty^\infty$-realisations grows like $\mathcal{W}^* e^{\alpha^* t_N}$ for $N \to \infty$, with $t_N \to \infty, t_N = o(\log N)$ for some non-degenerate random variable $\mathcal{W}^*$ and a number $\alpha^* \in (0,\infty)$. And finally we have to identify α^* as α.

To verify that, we have to establish either that for some suitable finite and not identically zero random variable $\mathcal{W}^*$ and number α^* we have

$$(1.68) \qquad \widehat{x}_2^N(t_N) := \sum_{i=1}^N x_2^N(i,t_N) \sim \mathcal{W}^* e^{\alpha^* t_N}$$

as $N \to \infty$, or alternatively that we need a spatial window of size $Ne^{-\alpha^* t_N}$ to find such a type-2 colonised site with nontrivial probability in that window.

In order to explain the origin of the randomness in $\mathcal{W}^*$ we need the behaviour of the law of the random variable

$$(1.69) \qquad \sum_{i=1}^N x_2^N(i,t) = \beth_t^{N,m}([0,1]) \,, \qquad t \in [0,T]$$

and the localization of this total mass on different sites for $N \to \infty$ in a suitable description, which is given by the full atomic measure $(\beth_t^{N,m})_{t \geq 0}$.

In particular if we have established the above scenario, we can conclude that it requires time $(\alpha^*)^{-1} \log N + t$ for some appropriate $t \in \mathbb{R}$ to reach intensity ε of type 2 in the whole collection of sites and then we can conclude that $\alpha^* = \alpha$.

We now want to show three things to describe the droplet growth: (1) There is a limiting law for the size of the type-2 population in a typical colonized site. (2) Identify the exponential growth rate of the number of colonized type-2 sites. (3) Identify the limiting dynamic as $N \to \infty$ of $(\beth_t^{N,m})_{t \geq 0}$ exactly as the dynamic $(\beth_t^m)_{t \geq 0}$.

Addressing first point (1), (2) (for (3) see below), we shall prove the following on droplet formation. We use here the notational convention

$$(1.70) \qquad a_N = \widehat{o}(bf(N)) \text{ if } \limsup_{N \to \infty} (a_N / f(N)) < b.$$

Proposition 1.9 *(Microscopic emergence and evolution: droplet formation)*

a) The Palm distribution stabilizes, i.e.

$$(1.71) \qquad \widehat{\mu}_{t_N}^N \underset{N\to\infty}{\Longrightarrow} \widehat{\mu}_\infty^\infty \text{ for } t_N \uparrow \infty \text{ with } t_N = \widehat{o}(\alpha^{-1} \log N)$$

and the limit has the property

$$(1.72) \qquad \widehat{\mu}^{\infty}_{\infty}((0,1)) = 1.$$

The law $\widehat{\mu}^{\infty}_{\infty}$ will be identified in terms of the excursion measure in (3.635).

b) The total type-2 mass $\gimel^{N,m}_{t_N}([0,1])$ grows at exponential rate α^*, i.e. (1.68) holds and

$$(1.73) \qquad \alpha^* = \alpha, \ \text{with } \alpha \text{ from Proposition 1.7.}$$

Furthermore we have a random growth factor:

$$(1.74) \qquad \mathcal{L}\left[\widehat{x}^N_2(t_N)e^{-\alpha t_N}\right] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\mathcal{W}^*], \qquad \text{for } t_N \uparrow \infty \text{ with } t_N = \widehat{o}(\alpha^{-1}\log N).$$

The random variable $\mathcal{W}^*$ is non-degenerate. □

Remark 5 *This result is obtained by showing that for sufficiently large t_1 and t_2:*

$$(1.75) \qquad \lim_{N\to\infty} P\left(|e^{-\alpha t_1}\widehat{x}^N_2(t_1) - e^{\alpha t_2}\widehat{x}^N_2(t_2)| > \varepsilon\right) < \varepsilon.$$

Remark 6 *The mutant population making up the total population at time $\alpha^{-1}\log N + t$ of emergence dates back to a rare mutant ancestor which appeared at a time in $[0, T(\varepsilon)]$ with probability at least $1-\varepsilon$ and $T(\varepsilon) < \infty$ for every $\varepsilon > 0$.*

Remark 7 *Growth in the spatial Fisher-Wright and the spatial branching model are different in nature. In the branching model we have growth of the total type-2 mass at exponential rate s, which is due to essentially a random number of families growing at that rate and hence with few sites with a large population. In the Fisher-Wright case we get an exponentially (rate $\alpha < s$) growing number of sites with a type-2 population of at least ε. Macroscopic emergence occurs once this growing droplet has volume const $\cdot N$.*

In order to understand the structure of the quantity $\mathcal{W}^*$ arising from (1.74), we have to investigate the behaviour also in the earlier time horizon $t \in [0, T_0]$ of $\gimel^{N,m}_t$ as $N \to \infty$. And addressing our point (3) from above (1.70) we obtain the following convergence result.

Proposition 1.10 *(Limiting droplet dynamic)*

As $N \to \infty$

$$(1.76) \qquad \mathcal{L}[(\gimel^{N,m}_t)_{t\geq 0}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[(\gimel^m_t)_{t\geq 0}]$$

in the sense of convergence of continuous $\mathcal{M}_a([0,1])$-valued processes where $\mathcal{M}_a([0,1])$ is equipped with the weak atomic topology. □

In the growth behaviour of the limit dynamics as $N \to \infty$ we obtained above, we recover the quantity α^* and $\mathcal{W}^*$ in the longtime behaviour of the limit dynamic $(\gimel^m_t)_{t\geq 0}$ in the following proposition.

Proposition 1.11 *(Long-time growth behaviour of $\gimel^m_t$)*

Assume that either $m > 0$ or $\gimel^0_0([0,1]) > 0$. Then the following growth behaviour of $\gimel^m$ holds.

(a) There exists α^* such that the following limit exists

$$(1.77) \qquad \lim_{t\to\infty} e^{-\alpha^* t} E[\gimel^m_t([0,1])] \in (0,\infty).$$

We have

$$(1.78) \qquad \alpha^* = \alpha,$$

where α is defined as in Proposition 1.7 (see also 3.139).

(b) Recall (1.74) for the law of $\mathcal{W}^*$. Then:

$$e^{-\alpha t}\beth^m_t([0,1]) \underset{t\to\infty}{\Longrightarrow} \mathcal{W}^* \text{ in probability.} \tag{1.79}$$

(c) The growth factor in the exponential is truly random:

$$0 < Var[\lim_{t\to\infty} e^{-\alpha t}\beth^m_t([0,1])] < \infty. \qquad \square \tag{1.80}$$

Remark 8 *We shall see in the proof that the random variable $\mathcal{W}^*$ reflects the growth of $\beth^m_t([0,1])$ in the beginning, as is the case in a supercritical branching process and hence $\mathcal{E}^* = \alpha^{-1}\log\mathcal{W}^*$ can be viewed as the random time shift of that exponential $e^{\alpha t}$ which matches the total mass of $\beth^m_t$ for large t.*

Given $\beth^m_t$, let $\nu_t(a,b]$, $0 < a < b \le 1$ denote the number of atoms in the interval $(a,b]$. Then the analogue of the size distribution for the discrete CMJ process is given by

$$\frac{\nu_t(\cdot)}{\beth^m_t([0,1])} \in \mathcal{P}((0,1])). \tag{1.81}$$

It is reasonable to expect that there is a limiting stable size distribution in the $t \to \infty$ limit similar to that in the discrete case but we do not follow-up on this here.

Part 3: Fixation

We now understand the preemergence situation and the time of emergence. In order to continue the program to describe the dynamics of macroscopic fixation, we consider the limiting distributions of the empirical measure-valued processes in a second time scale $\alpha^{-1}\log N + t, \quad t \in \mathbb{R}$. Define

$$\Xi^{\log,\alpha}_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{(x^N_1(i,\frac{\log N}{\alpha}+t),x^N_2(i,\frac{\log N}{\alpha}+t))}, \quad t \in \mathbb{R}, \quad (\Xi^{\log,\alpha}_N(t) \in \mathcal{P}(\mathcal{P}(\mathbb{K}))) \tag{1.82}$$

and then the two empirical marginals are given as

$$\Xi^{\log,\alpha}_N(t,\ell) := \frac{1}{N}\sum_{i=1}^N \delta_{x^N_\ell(i,\frac{\log N}{\alpha}+t)}, \ \ell = 1,2 \text{ and } t \in \mathbb{R}, \quad (\Xi^{\log,\alpha}_N(t,\ell) \in \mathcal{P}([0,1])). \tag{1.83}$$

Note that for each t and given ℓ the latter is a random measure on $[0,1]$. Furthermore we have the representation of the empirical mean of type 2 as follows:

$$\bar{x}^N_2(\frac{\log N}{\alpha} + t) = \int_{[0,1]} x\ \Xi^{\log,\alpha}_N(t,2)(dx). \tag{1.84}$$

The third main result of this section is on the fixation process in type 2, saying that $\Xi^{\log,\alpha}_N$ converges as $N \to \infty$ and the limit can be explicitly identified as a random McKean-Vlasov entrance law starting from time $-\infty$ in type 1.

Proposition 1.12 *(Asymptotic macroscopic fixation process)*

(a) For each $-\infty < t < \infty$ the empirical measures converge weakly to a random measure:

$$\mathcal{L}[\{\Xi^{\log,\alpha}_N(t,\ell)\}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\{\mathcal{L}_t(\ell)\}] = P^\ell_t \in \mathcal{P}(\mathcal{P}([0,1])), \text{ for } \ell = 1,2. \tag{1.85}$$

In addition we have path convergence:

$$w - \lim_{N\to\infty} \mathcal{L}[(\Xi^{\log,\alpha}_N(t))_{t\in\mathbb{R}}] = P \in \mathcal{P}[C((-\infty,\infty),\mathcal{P}(\mathcal{P}(\mathbb{K})))]. \tag{1.86}$$

A realization of P is denoted $(\mathcal{L}_t)_{t\in\mathbb{R}}$ respectively its marginal processes $(\mathcal{L}_t(1))_{t\in\mathbb{R}}, (\mathcal{L}_t(2))_{t\in\mathbb{R}}$.

(b) The laws $P_t^\ell, \ell = 1,2$ are non-degenerate probability measures on the space $\mathcal{P}([0,1])$. In particular

$$E\left[\left(\int_{[0,1]} x\mathcal{L}_t(2)(dx)\right)^2\right] > \left(E[\int_{[0,1]} x\mathcal{L}_t(2)(dx)]\right)^2. \tag{1.87}$$

(c) The process $(\mathcal{L}_t)_{t\in\mathbb{R}}$ describes the emergence and fixation dynamics, that is, for $t \in \mathbb{R}$, and $\varepsilon > 0$,

$$\lim_{t\to-\infty} Prob[\mathcal{L}_t(2)((\varepsilon,1]) > \varepsilon] = 0, \tag{1.88}$$

$$\lim_{t\to\infty} Prob[\mathcal{L}_t(2)([1-\varepsilon,1]) < 1-\varepsilon] = 0, \tag{1.89}$$

with

$$\mathcal{L}_t(2)((0,1)) > 0 \quad , \quad \forall t \in \mathbb{R},\ a.s.. \tag{1.90}$$

(d) For every $t \in \mathbb{R}$, always both type 1 and type 2 are present:

$$Prob[\mathcal{L}_t(2)(\{0\}) = 0)] = 1. \tag{1.91}$$

(e) The limiting dynamic in (1.86) is identified as follows:

The probability measure P in (1.86) is such that the canonical process is a random solution (recall Definition (1.2)) and entrance law from time $-\infty$ to the McKean-Vlasov equation (1.18).

(f) The limiting dynamic in (1.86) satisfies with α as in (1.64):

$$\mathcal{L}[e^{\alpha|t|}\int_{[0,1]} x\mathcal{L}_t(2)(dx)] \Rightarrow \mathcal{L}[^*\mathcal{W}] \text{ as } t \to -\infty, \tag{1.92}$$

and we explicitly identify the random element generating P in (1.86), namely P arises from random shift of a deterministic path:

$$P = \mathcal{L}[\tau_{^*\mathcal{E}}\mathcal{L}^*] \quad , \quad ^*\mathcal{E} = (\log^* \mathcal{W})/\alpha, \quad \tau_r \text{ is the time-shift of path by } r, \tag{1.93}$$

where $\mathcal{L}^*$ is the unique and deterministic entrance law of the McKean-Vlasov equation (1.18) satisfying (1.29) and with projection $\widetilde{\mathcal{L}}_t(2)$ on the type 2 coordinate satisfying:

$$e^{\alpha|t|}\int_{[0,1]} x\widetilde{\mathcal{L}}_t(2)(dx) \longrightarrow 1, \text{ as } t \to -\infty. \tag{1.94}$$

The random variable $^*\mathcal{W}$ satisfies

$$0 < {}^*\mathcal{W} < \infty \text{ a.s.}, \quad E[^*\mathcal{W}] < \infty, \quad 0 < Var(^*\mathcal{W}) < \infty. \tag{1.95}$$

(g) We have for $s_N \to \infty$ with $s_N = o(\log N)$ the approximation property for the growth behaviour of the limit dynamic by the finite N model, namely:

$$\mathcal{L}[e^{\alpha s_N}\bar{x}_2^N(\frac{\log N}{\alpha} - s_N)] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[^*\mathcal{W}]. \qquad \square \tag{1.96}$$

Corollary 1.13 *(Scaled total mass process convergence)*

The above implies for the total mass process that:

$$\mathcal{L}[\{\bar{x}_2^N(\frac{\log N}{\alpha} + t)\}_{t\in\mathbb{R}}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\{\int_{[0,1]} x\mathcal{L}_t(2)(dx)\}_{t\in\mathbb{R}}], \tag{1.97}$$

and we have convergence in distribution on path space. $\square$

Remark 9 *The random variable $\mathcal{W}^*$ of (1.74) can be viewed as the* final value *of the microscopic development of the mutant droplets and ${}^*\mathcal{W}$ of (1.92) can be viewed as an* entrance value *of the macroscopic emergence of the mutant type.*

This result of course immediately raises the question how the macroscopic *entrance* from 0 is related to the *final* value in the smaller original time scale. We have already stated that the two exponential growth rates from these two directions are the same. But more is true also the two growth constants have the same law:

Proposition 1.14 *(Relation between microscopic and macroscopic regimes)*
The entrance value and the final value are equal in law:

$$\mathcal{L}[\mathcal{W}^*] = \mathcal{L}[{}^*\mathcal{W}]. \qquad \square \tag{1.98}$$

Remark 10 *We note that this result cannot be derived from some soft argument since we relate two time scales t and $\alpha^{-1}\log N + t$ which separate as $N \to \infty$ and indeed the limits $N \to \infty$ and $t \to \infty$ or $t \to -\infty$ cannot be interchanged in general.*

Remark 11 *One could go further studying the relation between final and entrance configuration and ask how the limiting Palm distribution $\widehat{\mu}_\infty^\infty$, which describes the typical sparse type-2 sites is related to the limit $\mathcal{L}_{-\infty}$ as $t \to -\infty$ of the Palm distribution $\widehat{\mathcal{L}}_t$ of $\mathcal{L}_t$, i.e.*

$$\widehat{\mathcal{L}}_t(2)(dx) = \frac{x\mathcal{L}_t(2)(dx)}{\int x\mathcal{L}_t(2)(dx)}, \tag{1.99}$$

which describes the typical site "immediately" after emergence.

Note that this new transformed measure is now deterministic in the $t \to -\infty$ limit, since in the ratio the same random time shift is used in both numerator and denominator.

Here we have to observe that due to the exponential growth of type-2 sites the state depends on when they were colonised and that time point is typically a finite *random time back. Define for the process given in (1.33)*

$$\mu_s^*(dx) = \text{ probability that an excursion of age } s \text{ has size in } dx. \tag{1.100}$$

Note that for any time the total mass $\beth_t([0,1])$ is finite but there are countably many non-zero atoms in $\beth_t$. At time t let

$$\Gamma_t([a,b)) = \text{ number of atoms of age in } [a,b), \quad 0 < a < b \le 1 \tag{1.101}$$

and then define the corresponding size-biased distribution on the product space $\mathbb{R}^+ \times [0,1]$ of age and size given by

$$\widehat{\Gamma}_t(ds,dx) = \frac{\Gamma_t(ds)x\mu_s^*(dx)}{\int_0^t \Gamma_t(ds)\int_0^1 x\mu_s^*(dx)ds}. \tag{1.102}$$

By analogy with the Crump-Mode-Jagers theory, we would expect that this age and size distribution $\widehat{\Gamma}_t(ds,dx)$ stabilizes as $t \to -\infty$ with stable age and size distribution $\widehat{\Gamma}(ds,dx)$.

Then we would expect to have the relation:

$$\widehat{\mathcal{L}}_{-\infty}(2)(dx) = \widehat{\Gamma}((0,\infty),dx). \tag{1.103}$$

Corresponding to the limit dynamics of Proposition 1.12 we also have a *stochastic propagation of chaos result*, which characterizes the asymptotic behaviour of a tagged sample of sites as follows:

Proposition 1.15 *(Fixation dynamic for tagged sites)*
There is a unique limiting entrance law for the process of tagged components: for fixed $L \in \mathbb{N}$, the sequence in N of laws

$$\mathcal{L}[(\{x_2^N(i, \frac{\log N}{\alpha} + t))\}_{i=1,\dots,L})_{t\in\mathbb{R}}] \tag{1.104}$$

converges as $N \to \infty$ weakly to a law

$$\mathcal{L}[(X_2(t))_{t\geq 0}] = \mathcal{L}[((x_2(1,t),\cdots,x_2(L,t))_{t\in\mathbb{R}}], \text{ with } X_2(t) \longrightarrow \underline{0} \text{ as } t \to -\infty, \tag{1.105}$$

which is if restricted to $t \geq t_0$ for every $t_0 \in \mathbb{R}$ the weak solution of the SSDE:

$$\begin{aligned} dx_2(i,t) &= c\left(\int_{[0,1]} x\, \mathcal{L}_t(2)(dx) - x_2(i,t)\right)dt + sx_2(i,t)(1-x_2(i,t))dt \\ &+ \sqrt{d\cdot x_2(i,t)(1-x_2(i,t))}\, dw_i(t), \quad i = 1,\dots,L, \end{aligned} \tag{1.106}$$

where $w_1,\dots,w_L$ are independent standard Brownian motions which are also independent of the process $\{\mathcal{L}_t\}_{t\in\mathbb{R}}$ and the latter is given by (1.93). The behaviour at $-\infty$ in (1.105) and the equation (1.106) uniquely determine the law for a given mean curve. □

Remark 12 *Note that $x_2(i,t)$ will hit 0 at random times until $\bar{x}_2(t)$ exceeds a certain level depending on the other parameters in particular c/d, only after that time where $\bar{x}_2(t)$ remains large enough the component remains a.s. occupied with positive type-2 mass for all future time. Before that time the path will have zeros at an uncountable set of time points with Lebesgue measure zero, the set of these zeros will be stochastically bounded from above. This is a consequence of standard diffusion theory (see [RW]).*

2 Duality

The key tool to obtain our results is *duality* between our process and a particle system.

A duality relation for a Markov process Z with state space E is given as follows. There is a process Z' with state space E' such that for a function $H : E \times E' \to \mathbb{R}$, which is bounded and continuous we have

$$E_{Z_0}[H(Z_t, Z'_0)] = E_{Z'_0}[H(Z_0, Z'_t], t \geq 0. \tag{2.1}$$

The duality we use is a special case of the duality theory developed in [DGsel] and [DGDuality]. We begin constructing the dual particle system and then state the duality.

2.1 The dual process

We construct now a dual process. This dual process is based on a particle system $(\eta_t)_{t\geq 0}$, whose dependence on N we suppress in the notation.

We need the following ingredients to define the process η:

- N_t is a non-decreasing $\mathbb{N}$-valued process. N_t is the number of *individuals* present in the dual process with $N_0 = n$, the number of initially tagged individuals, and with $N_t - N_0 \geq 0$ given by the number of individuals born during the interval $(0,t]$.
- $\zeta_t = \{1,\cdots,N_t\}$ is an *ordered particle system* where the individuals are given an assigned order and the remaining particles are ordered by time of birth.

- $\eta_t =$ (ζ_t, π_t, ξ_t) is a trivariate process consisting of the above ζ and
  $-\pi_t$: *partition* $(\pi_t^1, \cdots, \pi_t^{|\pi_t|})$ of ζ_t, i.e. an ordered family of subsets, where the *index* of a partition element is the smallest element (in the ordering of individuals) of the partition element.
  $-\xi_t : \pi_t \longrightarrow \Omega_N^{|\pi_t|}$, giving *locations* of the partition elements. Here ξ_0 is the vector of the prescribed space points where the initially tagged particles sit.

- $\mathcal{F}_t$ is for given $\eta_t = (\zeta_t, \pi_t, \xi_t)$ is a function in $L_\infty(\mathbb{I}^{|\pi_t|})$.

Therefore the state of η is an element in the set

$$\mathcal{S} = \bigcup_{m=1}^{\infty} \{1, 2, \cdots, m\} \times \text{Part}^{<}(\{1, \cdots, m\}) \times \{\xi : \text{Part}^{<}(\{1, \cdots, m\}) \longrightarrow \Omega_N\}, \tag{2.2}$$

where $\text{Part}^{<}(A)$ denotes the set of ordered partitions of a set $A \subseteq \mathbb{N}$ and the set $\{1, \cdots, m\}$ is equipped with the natural order.

It is often convenient to order the partition elements according to their indices

$$\textit{index } (\pi_t^i) = min(k \in \mathbb{N} : k \in \pi_t^i) \tag{2.3}$$

and then assign them ordered

$$\textit{labels } 1, 2, 3, \cdots, |\pi_t|. \tag{2.4}$$

Remark 13 *Note that*

$$N_t = |\pi_t^1| + \cdots + |\pi_t^{|\pi_t|}|. \tag{2.5}$$

Remark 14 *The process η can actually be defined starting with countably many individuals located in Ω_N. This is due to the quadratic death rate mechanism at each site which implies that the number of individuals at any site will have jumped down into $\mathbb{N}$ by time t for any $t > 0$.*

The dynamics of $\{\eta_t\}$ is that of a pure Markov jump process with the following transition mechanisms which correspond to resampling, migration and selection in the original model (in this order):

- *Coalescence* of partition elements: any pair of partition elements which are at the same site coalesce at a constant rate; the resulting enlarged partition element is given the smaller of the indices of the coalescing elements as index and the remaining elements are reordered to preserve their original order.

- *Migration* of partition elements: for $j \in \{1, 2, \ldots, |\pi_t|\}$, the partition element j can migrate after an exponential waiting time to another site in Ω_N which means we have a jump of $\xi_t(j)$.

- *Birth* of new individuals by each partition element after an exponential waiting time: if there are currently N_t individuals then a newly born individual is given index $N_{t-} + 1$ and it forms a *partition element consisting of one individual* and this partition element is given a label $|\pi_{t-}| + 1$. Its location mark is the same as that of its parent individual at the time of birth,

- *Independence*: all the transitions and waiting times described in the previous points occur independently of each other.

Thus the process $(\eta_t)_{t\geq 0}$ has the form

$$\eta_t = (\zeta_t,\ \pi_t, (\xi_t(1), \xi_t(2), \cdots, \xi_t(|\pi_t|))), \tag{2.6}$$

where $\xi_t(j) \in \Omega_N,\ j = 1, \ldots, |\pi_t|$, and π_t is an ordered partition (ordered tuple of subsets) of the current basic set of the form $\{1, 2, \ldots, N_t\}$ where N_t also grows as a random process, more precisely as a pure birth process.

We order partition elements by their smallest elements and then *label* them by $1, 2, 3, \ldots$ in increasing order. Every partition element (i.e. the one with label i) has at every time a location in Ω_N, namely the i-th partition element has location $\xi_t(i)$. (The interpretation is that we have N_t individuals which are grouped in $|\pi_t|$-subsets and each of these subsets has a position in Ω_N).

Furthermore denote by

$$\pi_t(1), \cdots, \pi_t(|\pi_t|) \tag{2.7}$$

the index of the first, second etc. partition element. In other words the map gives the index of a partition element (the smallest individual number it contains) of a label which specifies its current rank in the order.

For our concrete situation we need to specify in addition the parameters appearing in the above description, this means that we define the dual as follows.

Definition 2.1 *(The dual particle system: η_t)*
(a) The initial state η_0 is of the form (ζ_0, π_0, ξ_0) with:

$$\zeta_0 = \{1, 2, \cdots, n\}, \quad \pi_0 = \{\{1\}, \cdots, \{n\}\} \tag{2.8}$$

$$\xi_0 \in (\Omega_N)^n. \tag{2.9}$$

(b) The evolution of (η_t) is defined as follows: u

(i) each pair of partition elements which occupy the same site in Ω_N coalesces *during the joint occupancy of a location into one partition element after a* rate d *exponential waiting time,*

(ii) every partition element performs, independent of the other partition elements, a continuous time random walk *on Ω_N with transition* rates $c\bar{a}(\cdot, \cdot)$, *with $\bar{a}(\xi, \xi') = a(\xi', \xi)$,*

(iii) after a rate s exponential waiting time each partition element gives birth *to a new particle which forms a new (single particle) partition element at its location, and this new partition element is given as label $|\pi_{t-}| + 1$ and the new particle the index $N_{t-} + 1$.*

All the above exponential waiting times are independent *of each other.* □

Note that this process is well-defined since the total number of partition elements is stochastically dominated by an ordinary linear birth process which is well-defined for all times.

2.2 The duality relation

This gives in our case (see [DGDuality]) a duality relation between X^N given in (1.1)and η given by (2.6).

The duality relation reads in our case for $i_1, \cdots, i_n \in \{1, \cdots, N\}$ all different:

$$E[\prod_{k=1}^{n} x_{i_k}^N(t)] = E[exp(-\frac{m}{N} \int_0^t \Pi_u^{N,n,1} du)], \tag{2.10}$$

where $\Pi_u^{N(n,1)}$ denotes the number of dual particles at time u starting with the initial state (2.8), (2.9).

We also need the duality relation for the *McKean Vlasov process* $\{(x_\ell^\infty(0,t), x_\ell^\infty(*,t)), \quad \ell = 1,2\}$ which is given by $x_1 = 1 - x_2$ and

$$dx_2^\infty(0,t) = c(\theta - x_2^\infty(0,t))dt \qquad +s(x_2^\infty(0,t)(1-x_2^\infty(0,t))dt + \sqrt{d \cdot x_2^\infty(0,t)(1-x_2^\infty(0,t)}dw(t) \tag{2.11}$$

$$x_2^\infty(*,t) \equiv \theta. \tag{2.12}$$

The dual process for the McKean-Vlasov processes arises if we replace

$$\{1, \cdots, N\} \text{ by } \{0, *\} \tag{2.13}$$

such that all rates of the dual particle on $*$ are put equal to zero and the jump at rate c from 0 is always to $\{*\}$. Otherwise the system remains.

The duality relation now reads:

$$E[(x_2^\infty(t,0))^k] = E[\theta^{|\Pi_t^{\infty,k}|}]. \tag{2.14}$$

3 Proof of results

We begin in Section 2 to state the duality theory and continue in Subsubsections 3.1-3.6 with the proofs. We start in Subsection 1.19 with a warm-up exhibiting some of the methods in simpler cases. (This is not needed for the rigorous argument). Then we continue with the formal proofs, namely proving

- in Subsection 3.2 assertions related to the large time scale, in particular Proposition 1.3, Proposition 1.7 and Proposition 1.12,
- in Subsection 3.3 assertions concerning the small time scale and droplet formations, in particular Proposition 1.5, Proposition 1.9 and Proposition 1.10,
- in Subsections 3.4 and 3.5 assertions concerning the connection between the two time scales, in particular Proposition 1.14,
- in Subsections 3.6-3.7 additional assertions where different methods are used, in particular Proposition 1.15 in Subsection 3.6 and extensions in Subsection 3.7.

3.1 A warm-up: The case of a single site model

Before we start proving all the statements made in Subsection 1.3 we look at some simpler cases and already give in the arguments a flavour of the methods used later on. This subsection may be skipped from a purely logical point of view. These simpler systems are single site models.

In the *absence of migration*, $c = 0$, the individual sites evolve independently. In this case it suffices to study a system

$$X(t) = (x_1(t), x_2(t)) \tag{3.1}$$

at a single isolated site, for example use $x_1(t) = x_1(1,t), x_2(t) = x_2(1,t)$. Recall (1.1-1.6).

We begin by showing in this case that emergence in $\log N$ time scale does occur in the infinite population, that is, deterministic case, i.e. $d = 0$ but *does not* occur in the finite population case,

that is, random case, where $d > 0$. This demonstrates the effect of the bounded component (mass restriction) and the important role played by space and migration.

The proof of this result gives us the opportunity to introduce some basic ideas and methods in a simple setting. We consider here successively the deterministic and stochastic case and collect the proofs with the dual calculations in the third Subsubsection 3.1.3.

3.1.1 Emergence of rare mutants in the single site deterministic population model

We first consider the deterministic, i.e. $d = 0$ (often called infinite population) model without migration, i.e. $c = 0$. In the case of small mutation rate, $\frac{m}{N}$, we will show with two different methods that asymptotically as $N \to \infty$ emergence occurs at times $\frac{\log N}{s} + t, t \in \mathbb{R}$ with fixation as $t \to \infty$:

Lemma 3.1 *Assume that the mutation rate is* $\frac{m}{N}$*, and* $d = 0,\ s > 0,\ c = 0$*. Then we have:*

$$\lim_{N\to\infty} x_2\left(\frac{\log N}{s} + t\right) = 1 - \frac{s}{s + me^{st}} \begin{cases} \to 1, & as \quad t \to \infty \\ \to 0, & as \quad t \to -\infty. \end{cases} \qquad \square \tag{3.2}$$

Remark 15 *This means that in the deterministic case we have emergence and fixation in the "critical" time scale* $(s^{-1} log\ N) + t$ *as in the branching approximation.*

3.1.2 Emergence of rare mutants in the single site stochastic population model

The above analysis can easily be extended to the stochastic case (often called finite population), that is, $d > 0$, but now we get a qualitatively different conclusion. We again take mutation rate $\frac{m}{N}$. Namely we observe that

Lemma 3.2 *Let* $d > 0$ *and* $c = 0$*, then*

$$\mathcal{L}[x_2^N(a_N)] \underset{N\to\infty}{\Longrightarrow} \delta_0 \text{ whenever } a_N/N \to 0 \text{ as } N \to \infty. \tag{3.3}$$

Therefore emergence and takeover does not occur in time scale $\log N$ *if there is no migration, that is,* $c = 0$. $\square$

Remark 16 *This means that the "finite population sampling", i.e.* $d > 0$*, seriously slows down the emergence of rare advantageous mutants. This agrees with the corresponding result in the branching case where the fluctuation term is responsible for the emergence on* $O(N)$ *scale if* $c = 0$*. This effect is removed as soon as we have migration, which is therefore the key mechanism for the emergence if* $d > 0$.

3.1.3 Proofs using the dual

Proof of Lemma 3.1. The Lemma 3.1 is now proved in two ways with and without duality. We consider first an auxiliary object $x_2(t)$ for which if we consider the case where the parameter m is replaced by $\frac{m}{N}$ we get our x_2^N.

(i) Analytic approach The quantity $x_2(t)$ satisfies the ODE

$$\frac{dx_2}{dt} = m(1 - x_2) + sx_2(1 - x_2); \quad x(0) = 0 \tag{3.4}$$

with solution

$$x_2(t) = \frac{m(e^{(m+s)t} - 1)}{s + me^{(m+s)t}}. \tag{3.5}$$

We can now study this explicit solution to the ODE for our choices of the parameters and the appropriate time scales where t is replaced by $\frac{1}{s}\log N + t$. (See point (iii) below).

(ii) Duality approach. As a prelude to the analysis of the stochastic case we show how the formula (3.5) can be obtained using the *dual representation*.

We assume that $Y_0 = \delta_1$, $s > 0$ and we choose $f_0 = 1_{\{1\}}$. The mutation part of the dual consists of the simple transition corresponding to a mutation from type 1 to type 2. This means for the dual that we have jumps $1_{\{2\}} \to 1$, $1_{\{1\}} \to 0$ both at rate m.

Let $\{p_k(t)\}_{k\in\mathbb{Z}_+}$ be the distribution of a pure birth process N_t starting with one individual (and each individual independently gives birth at rate s at time t) and let

$$\tau = \text{ time of first mutation jump.} \tag{3.6}$$

Let $\tilde{p}_k(t)$ denote the probability that there were $(k-1)$ births in $[0,t]$ and that no mutation jump has occurred.

Then the random state of ${}^{FK}\mathcal{F}_t^+$ at time t is given by:

$${}^{FK}\mathcal{F}_t^+ = (1_{\{1\}})^{\otimes k} 1_{\{\tau > t\}}, \tag{3.7}$$

with probability $\tilde{p}_k(t)$ and 0 with probability $1 - \sum_{k=1}^{\infty} \tilde{p}_k(t)$. Then

$$x_1(t) = E[< X(t), 1_{\{1\}} >] = E[\int {}^{FK}\mathcal{F}_0^+ dX_t] = E[\int {}^{FK}\mathcal{F}_t^+ dX_0^{\otimes|\pi_t|}] = P[\tau > t]. \tag{3.8}$$

We have:

$$P(\tau > t) = \sum_{k=1}^{\infty} \tilde{p}_k(t). \tag{3.9}$$

Noting that the rate of a jump to zero is mn when there are n factors it follows that the collection $\{\tilde{p}_n(t), n \in \mathbb{N}\}$ satisfies the system of ODE's:

$$\begin{aligned} &\frac{d\tilde{p}_n(t)}{dt} = (n-1)s\ \tilde{p}_{n-1}(t) - n(s+m)\tilde{p}_n(t) \\ &\tilde{p}_1(0) = 1. \end{aligned} \tag{3.10}$$

To determine the $\{\tilde{p}_k(t), \quad k \in \mathbb{N}\}$ we introduce the Laplace transform:

$$R(\theta, t) = \sum_{n=1}^{\infty} e^{n\theta} \tilde{p}_n(t). \tag{3.11}$$

Then using (3.10) it can be verified that $R(\cdot,\cdot)$ satisfies

$$\begin{aligned} R(\theta,0) &= e^{\theta} \\ \frac{dR(\theta,t)}{dt} &= \sum_{n=1} [se^{n\theta}(n-1)\tilde{p}_{n-1}(t) - (s+m)\sum ne^{n\theta}\tilde{p}_n(t)] \\ &= se^{\theta}\frac{\partial}{\partial\theta}R(\theta,t) - (s+m)\frac{\partial}{\partial\theta}R(\theta,t) \\ &= (se^{\theta} - (s+m))\frac{\partial}{\partial\theta}R(\theta,t). \end{aligned} \tag{3.12}$$

The solution to this PDE is obtained by the method of characteristics as follows. In symbolic notation:

$$\frac{dt}{1} = \frac{d\theta}{-(se^{\theta} - (s+m))} = \frac{dR}{0}, \tag{3.13}$$

so that we obtain the characteristic curve

$$(3.14)\qquad e^{(s+m)t}(s-(s+m)e^{-\theta}) = \text{constant}.$$

Hence the general solution is

$$(3.15)\qquad R(\theta,t) = \Psi(e^{(s+m)t}(s-(s+m)e^{-\theta})),$$

where Ψ is an arbitrary function. Using the initial condition

$$(3.16)\qquad e^{\theta} = \Psi(s-(s+m)e^{-\theta})$$

we get that

$$(3.17)\qquad \Psi(u) = \frac{s+m}{s-u}.$$

It follows that

$$(3.18)\qquad R(\theta,t) = \frac{s+m}{s-e^{(s+m)t}(s-(s+m)e^{-\theta})}.$$

Therefore

$$(3.19)\qquad 1-x_2(t) = \sum_{n=1}^{\infty}\tilde{p}_n(t) = R(0,t) = \frac{s+m}{s+me^{(s+m)t}}.$$

(iii) Large N asymptotic. We now consider the emergence of a rare mutant with m replaced by $\frac{m}{N}$ and consider the time scale $(C\log N)+t, \quad t\in\mathbb{R}$. We get

$$(3.20)\qquad 1-x_2(C\log N+t) = \frac{s+\frac{m}{N}}{s+\frac{m}{N}e^{(s+\frac{m}{N})(C\log N+t)}} = \frac{s+\frac{m}{N}}{s+\frac{m}{N}e^{(s+\frac{m}{N})t}N^{C(s+\frac{m}{N})}},$$

so that $1-x_2(C\log N+t)\to 1$ or 0 for all $t\in\mathbb{R}$ depending on $C<\frac{1}{s}$ or $C>\frac{1}{s}$. In other words the rare mutant emerges and takes over in time scale $s^{-1}\log N$ as $N\to\infty$. We shall derive more precise results later in Subsection 3.3 for example in (3.647). Finally, for $C=\frac{1}{s}$,

$$(3.21)\qquad 1-x_2(C\log N+t) = \frac{s+\frac{m}{N}}{s+me^{(s+\frac{m}{N})t}N^{\frac{m}{sN}}} \to \frac{s}{s+me^{st}}, \quad \text{as } N\to\infty.$$

Proof of Lemma 3.2

We consider the dual process $(\eta_t^N, FK\mathcal{F}_t^+)_{t\geq 0}$ and define

$$(3.22)\qquad \tau_N = \inf\{t : ^{FK}\mathcal{F}_t^+ \equiv 0\}.$$

For the deterministic case, $d=0$, the random time τ_N was of order $s^{-1}\log N$, but this now changes to a much larger order of magnitude. This is made precise in the following Lemma that together with the dual identity completes the proof of Lemma 3.2.

Lemma 3.3 *Let $d>0$ and $c=0$.*

If $\lim\frac{a_N}{N}\to 0$, then

$$(3.23)\qquad P(\tau_N<a_N)\to 0 \textit{ as } N\to\infty \textit{ for all } t>0. \qquad \square$$

Proof of Lemma 3.3

Start the dual $\mathcal{F}^+$ in $1_{\{1\}}$. In this case up to the time of the first mutation the number of factors $1_{\{1\}}$ in (3.7) is equal to or greater than 1. Therefore for each N, $P(\tau_N < \infty) = 1$.

This implies that for every *fixed* N

$$x_2^N(t) \to 1, \text{ a.s. as } t \to \infty. \tag{3.24}$$

On the other hand the number of factors is stochastically bounded due to the coalescence (which occurs at quadratic rate as soon as $d > 0$). To verify this we can calculate the expected number of factors and show that it remains bounded by obtaining an upper bound given by the solution of an ODE as follows.

Consider the birth and death process $\Pi_t^{(n)} = \Pi_t^{N,(n,1)}$ representing the number of factors of which $\mathcal{F}_t^+$ consists of at time t if we are starting at $\Pi_0^{(n)} = n$ at one site. The dynamics of this object follows immediately from the evolution of $(\mathcal{F}_t^+)_{t\geq 0}$. The forward Kolmogorov equations of the process $\Pi_t^{(n)}$ are therefore given by

$$p'_{ij}(t) = s(j-1)p_{i,j-1}(t) + \frac{d}{2}(j+1)jp_{i,j+1}(t) - (sj + \frac{d}{2}j(j-1))p_{ij}(t), \tag{3.25}$$

$$p_{n,n}(0) = 1. \tag{3.26}$$

Letting $E[\Pi_t^{(n)}]^k = m_k(t) = \sum_{j=0}^{\infty} j^k p_{n,j}(t)$, we obtain with $\hat{s} = s + \frac{d}{2}$:

$$\frac{dm_1(t)}{dt} = \hat{s}m_1(t) - \frac{d}{2}m_2(t) \leq \hat{s}m_1(t) - \frac{d}{2}m_1^2(t), \quad m_1(0) = n. \tag{3.27}$$

Therefore

$$E[\Pi_t^{(n)}] \leq \frac{2\hat{s}ne^{\hat{s}t}}{nd(e^{\hat{s}t}-1) + 2\hat{s}}, \tag{3.28}$$

so that

$$E[\int_0^t \Pi_u^{(n)} du] \leq \frac{2}{d}\log(\frac{2\hat{s}}{d} + n(e^{\hat{s}t}-1)). \tag{3.29}$$

This implies that if we let $1 > \varepsilon > 0$ and choose α so that $1 - e^{-\alpha} = \varepsilon$, then using (3.29) we get:

$$P[\tau_N < a_N] \leq \varepsilon + P[\frac{m}{N}\int_0^{a_N} \Pi_s^{(n)} ds > \alpha] \leq \varepsilon + \frac{2m}{d\cdot\alpha}\frac{\text{const} + [\log n + \hat{s}a_N]}{N} \tag{3.30}$$

Hence $\lim_{N\to\infty} P(\tau_N < a_N) \leq \varepsilon$ for arbitrary $\varepsilon > 0$ and the proof is complete.

3.2 Proofs of Propositions 1.3, 1.7, and 1.12.

In this section we consider the large time scale $\alpha^{-1}\log N + t, \quad t \in \mathbb{R}$ and prove Proposition 1.3 on properties of the limit dynamics in this *emergence* and *fixation time scale* and the convergence results stated in Subsubsection 1.3.3, in particular, on the time needed for emergence and the form of the process of fixation, i.e. Propositions 1.7, 1.12. (Recall the remaining assertions requiring analysis in different time scales, in particular Propositions 1.9 and 1.11 on droplet formation are proved later in Subsection 3.3 and Proposition 1.15 on tagged sample convergence which uses different methods will be proved in Subsection 3.6.)

3.2.1 Outline of the strategy of proofs for the asymptotic analysis

The proofs for the large time scale proceeds in altogether thirteen subsubsections. Therefore we describe here in 3.2.1 first the general strategy and list at the end what happens in the various subsubsections.

The proofs of the results on the $N \to \infty$ asymptotics for the process in large time scales involve calculating the mean of $x_1^N(t)$ or $x_2^N(t)$ respectively of higher moments and hence depend in an essential way on an asymptotic analysis of the dual process which we introduced in Section 2.

In the case of two types the required dual calculation involves a system of particles (corresponding to factors) located at sites in $\{1, \ldots, N\}$ with linear birth rate and quadratic death rate at each site and migration between sites. Assuming that the initial population consists of only one individual and the initial function f is the indicator of type 1 at site 1, then we get a growing string of factors of such indicators, until the rare mutation operator (at rate $\frac{m}{N}$) acts on it at which time it and therefore the product becomes zero. Hence the main point is to prove that a mutation event somewhere in the string will occur in the $\log N$ time scale. For this to be the case we have to prove that the total number of factors reaches $O(N)$ in that time scale.

Consider now the dual particle system η starting with only finitely many particles. As long as the number of occupied sites remains $o(N)$ the migration of individuals leads in the limit $N \to \infty$ always to an unoccupied site. More precisely, in this regime the proportion of sites at which a collision occurs, that is a migration to a previously occupied site, is negligible. For that reason the collection of *occupied sites* can be viewed (asymptotically as $N \to \infty$) as a *Crump-Mode-Jagers branching process* (a notion we shall recall later on). This process grows exponentially and the identification of the emergence time corresponds to the Malthusian parameter α of this CMJ-branching process.

In order to investigate the dynamics in the next phase between emergence and fixation, i.e. in times $C \log N + t$ with $t \in \mathbb{R}$ the new time parameter, we must on the side of the dual particle system now consider the role of *collisions* of dual particles once the number of occupied sites is of order $O(N)$ and the proportion of occupied sites at which collisions occur is asymptotically non-zero. This can be handled by identifying a dynamical law of large numbers described by a *nonlinear* evolution equation. Nonlinear due to the immigration term at a site, resulting from immigration from other sites and which in the limit depends on their law. On the side of the McKean-Vlasov limit, this makes the evolution of the frequency of type 2 at single site depend on a global signal, namely the current spatial intensity of type 2.

The delicate issue of the *transition* between these two regimes with and without collision must be addressed to fully describe the emergence and to show that this leads to the McKean-Vlasov dynamics with random initial condition. For that purpose the system is considered in the time scale $\alpha^{-1} \log N + t$, with $t \in \mathbb{R}$ the macroscopic time parameter. The key element in the analysis is the determination of the leading term in an asymptotic expansion of the type-2 mass in time scale $\alpha^{-1} \log N + t$ with t varying, once we have taken $N \to \infty$, as $t \to -\infty$. On the side of the dual process this will correspond to an expansion of the normalised number of occupied sites as $N \to \infty$ and then $t \to \infty$ in the above time scale. However also in the discussion on droplet formation later on in subsection 3.3 it is necessary to keep track of the collisions since we need higher-order terms in the expansion of the total type-2 mass and these terms involve collisions of dual particles. This will be dealt with in Subsubsection 3.3.5 and the sequel.

To carry out this program we must study the behaviour of the dual particle system in two time regimes, namely,

(3.31) $\quad s(N) + t$ where $s(N) \to \infty$ but $s(N) = o(\log N)$

and for suitably chosen α,

$$\frac{\log N}{\alpha} + t, \quad -\infty < t < \infty, \tag{3.32}$$

and their asymptotics as $N \to \infty$. The first we call the *collision-free regime* or *pre-emergence regime*, the second the *collision regime*.

A key step in the dual process analysis is a coupling of the pre-emergence and collision regimes in order to understand the *transition* between the two regimes and to be able to consider the time scale

$$\frac{1}{\alpha} \log N + t_N \text{ with } t_N \to -\infty, \quad t_N = o(\log N). \tag{3.33}$$

Outline of Subsection 3.2.

This subsection focusses on the proofs of Proposition 1.3 (Subsubsection 3.2.1) on the limiting McKean-Vlasov dynamics, Proposition 1.7 (Subsubsections 3.2.1 -3.2.6) and Proposition 1.12 on emergence respectively convergence to the limiting dynamics of fixation (Subsubsections 3.2.7-3.2.13).

The Subsubsection 3.2.3 is devoted to the proof of Proposition 1.3, which specifies the limiting laws in short and large time scales.

In Subsubsections 3.2.4-3.2.6 we begin the proof of Proposition 1.7 on the order of magnitude of the emergence time. In a first step we exhibit the branching structure for the number of sites in 3.2.3 and then analyse the collision free, respectively the collision regime, 3.2.4 to 3.2.6.

The proof of Proposition 1.12 on the limit dynamics is more involved and is broken in six preparatory steps which are entirely concerned with the dual process at times $\frac{1}{\alpha} \log N + t$ and a final argument returning to our process, each of these is in one of the separate Subsubsections 3.2.7-3.2.13. Namely, we study in Subsubsections 3.2.7, 3.2.8, 3.2.9 and 3.2.10 the structure of the dual process more carefully, show in 3.2.11 convergence of the process to the limiting dynamics and examine in Subsubsection 3.2.12 the random growth constant in the emergence. Then we are able in Subsubsection 3.2.13 to complete the proof of Proposition 1.12 on the limiting dynamics of fixation.

3.2.2 Proof of Proposition 1.3 (Properties McKean-Vlasov dynamic)

In order to prove the results on the McKean-Vlasov dynamic recall the duality in Section 2. We now go through the different parts of the proposition to prove, but here the hard part is assertions d and e.

(a) This is the wellknown McKean-Vlasov limit for the interacting Fleming-Viot model with migration and selection (compare, for example, [DG99], Theorem 9).

(b) We want to show that $E[x_1(t)] \to 0$ as $t \to \infty$ if $E[x_1(0)] < 1$. This follows by a simple application of the mean-field dual $(\widetilde{\eta}_t, \mathcal{F}_t^+)_{t \geq 0}$ started with $\mathcal{F}_0^+ = 1_{\{1\}}$ and η_0 having one particle located at site 0. As time proceeds in the dual process the selection operator keeps being applied. But the ℓ-fold application of the selection operator, which now in the two-type case is simply obtained as $f \to f \otimes 1_{\{1\}}$, produces as surviving terms for $\mathcal{F}_t^+$ a string of the form (note coalescence only changes here the number of factors belonging to distinct variables)

$$\Pi_{i=1}^{\ell(t)} 1_{\{1\}}(u_{\xi_i(t)}) \tag{3.34}$$

and since the total number of factors diverges as soon as the migration rate c is positive, i.e. $\ell(t) \to \infty$, the result follows from the assumption $E[x_1(0)] < 1$.

(c) For existence it suffices to obtain a particular solution $(\mathcal{L}_t^*(2))_{t\in\mathbb{R}}$ where $\mathcal{L}_0^*(2) \in \mathcal{P}([0,1])$. Take the solution $(\widetilde{\mathcal{L}}_t^n(2))_{t\geq 0}$ with $\widetilde{\mathcal{L}}_0^n(2) = \mu_n$ with

$$(3.35) \qquad a_n := \int_0^1 x\mu_n(dx), a_n \to 0 \text{ as } n \to \infty.$$

In particular we can take here $\mu_n = \delta_{a_n}$. Choose now $(a_n)_{n\in\mathbb{N}}$ with $0 < a_n < \frac{1}{2}$ and $a_n \to 0$ as $n \to \infty$. Then first of all for every fixed n

$$(3.36) \qquad \lim_{t\to\infty} \widetilde{\mathcal{L}}_t^n(2) = \delta_1$$

and there exists $r(a_n)$ such that $\int\limits_0^1 x\widetilde{\mathcal{L}}_{r(a_n)}^n(2)dx = \frac{1}{2}$. In fact by duality we see that $r(a)$ is strictly increasing and continuous in a.

Then define

$$(3.37) \qquad (\mathcal{L}_t^n(2))_{t\geq -r(a_n)} = (\widetilde{\mathcal{L}}_{t+r(a_n)}^n(2))_{t\geq -r(a_n)}.$$

Since for any $t > 0$,

$$(3.38) \qquad \lim_{n\to\infty} \widetilde{\mathcal{L}}_t^n(2) = \delta_0,$$

it follows again by duality that $\lim_{n\to\infty} r(a_n) = \infty$. The value at $t = -r(a_n)$ tends to δ_0 as $n \to \infty$.

We note that if for a subsequence $(n_k)_{k\in\mathbb{N}}$ the sequence $(\mathcal{L}_s^{n_k})_{k\in\mathbb{N}}$ converges as $k \to \infty$ for some $s \in \mathbb{R}$, then it converges for all times $t \geq s$, as we immediately read off from the duality which gives the Feller property (continuity in the initial state) of the McKean-Vlasov process. Therefore by considering for example $s = -k, k \in \mathbb{N}$ we can obtain a further subsequence converging for all times $t \in \mathbb{R}$. Now we take such a convergent subsequence of $(\mathcal{L}^n(2))_{n\in\mathbb{N}}$ and define $\mathcal{L}^*(2)$ as the $(n \to \infty)$ limit. By construction $\int x\widetilde{\mathcal{L}}_0^n(2)(dx) = \frac{1}{2}$ and therefore $\mathcal{L}_t^*(2) \to \delta_1$ as $t \to \infty$. It remains therefore to argue that $\mathcal{L}^*(2)$ is a solution and is converging to δ_0 as $t \to -\infty$.

The fact that we obtain a solution of the equation is a consequence of the construction together with the continuity in the initial state which is immediate using the duality. Namely for every n we have a solution for $t \geq -\tau(a_n)$. Then using duality we see that the limit is a solution as well, since it satisfies the duality relation between times $s < t; s, t \in \mathbb{R}$.

The convergence to δ_0 as $t \to -\infty$ follows by contradiction as follows. Assume we converge to a strictly positive mean. Then applying the duality to calculate the mean at time 0 starting at times $t \downarrow -\infty$ we see, that this mean converges to one and therefore we obtain convergence to δ_1 (by (3.36)) at time 0. This contradicts the assumption that the mean is $\frac{1}{2}$ at time 0. This completes the proof of existence of a solution on $(-\infty, \infty)$ satisfying (1.26).

(d, e) Here we proceed in three steps, first we establish existence of solutions whose mean curve has the claimed growth behaviour at $-\infty$ and then we show uniqueness of the mean curve with this growth behaviour and subsequently the uniqueness of the solution for given mean curve.

Step 1 *Growth of mean-curves*

The first step is to consider the growth of solutions of the McKean-Vlasov equation, $\mathcal{L}_t(2)$, at $-\infty$. We will show two things,

- first there exists a solution with mean curve $m(\cdot)$ which behaves like $m(t) \sim Ae^{\alpha t}$ for $t \to -\infty$ with $A \in (0, \infty)$,
- secondly that all solutions satisfying the growth condition posed in (d) have mean curves satisfying $m(t) \sim Ae^{\alpha t}$ as $t \to -\infty$ for some $A \in [0, \infty)$.

In fact we shall combine these two points in our proof.

First assume that $(\mathcal{L}_t(2))_{t\in\mathbb{R}}$ is a solution (and entrance law) with mean-curve $(m(t))_{t\in\mathbb{R}}$ which satisfies:

$$m(t) > 0, \qquad 0 < \liminf_{t\to-\infty} e^{-\alpha t} m(t) < \infty. \tag{3.39}$$

We want to show that $e^{-\alpha t}m(t)$ converges to some $A \in \mathbb{R}^+$.

We can chose a sequence t_n with $t_n \to -\infty$ so that for some $A \in (0,\infty)$ we have

$$e^{\alpha|t_n|} m(t_n) = e^{\alpha|t_n|} \int_0^1 x \mathcal{L}_{t_n}(2)(dx) \to A \text{ as } n \to \infty. \tag{3.40}$$

We now abbreviate

$$\mu_{t_n} := \mathcal{L}_{t_n}(2). \tag{3.41}$$

What we know about μ_{t_n} is its mean and from this we have to draw all conclusions on the convergence, namely that the limit A does not depend on the choice of $(t_n)_{n\in\mathbb{N}}$.

On the other hand we can start with such a measure μ_{t_n} as in (3.41) and try to construct a solution with a mean-curve $(m(t))_{t\in\mathbb{R}}$, $\quad m(t) = (A + o(1))e^{\alpha t}$ as $t \to -\infty$ as the limit point of a sequence of solutions starting at time t_n in μ_{t_n}. A solution starting in μ_{t_n} at time t_n is denoted:

$$(\widetilde{\mathcal{L}}^n_t)_{t\geq t_n}. \tag{3.42}$$

Then we know, that if $(\mathcal{L}_t(2))_{t\in\mathbb{R}}$ is an entrance law with $\mathcal{L}_{t_n} = \mu_{t_n}$, then by uniqueness of the solution starting at t_n we must have

$$\mathcal{L}_t(2) = \widetilde{\mathcal{L}}^n_t(2), \text{ for } t \geq t_n. \tag{3.43}$$

Therefore our strategy now is twofold. First we verify that in fact the mean curve $(m(t))_{t\in\mathbb{R}}$ of $(\mathcal{L}_t(2))_{t\in\mathbb{R}}$ has the form:

$$m(t) = \exp(\alpha(t + \frac{1}{\alpha}\log A)) + o(e^{\alpha t}). \tag{3.44}$$

Then we argue that in fact the $(\widetilde{\mathcal{L}}^n_t)_{t\geq t_n}$ with mean curves satisfying (3.40) converge to an entrance law with mean curve satisfying (3.44). We begin with the first point and come to the second one (which is then simpler by what we have prepared for the first point) after (3.88).

In order to carry this program out we must consider the different possibilities we have for $\mu_{t_n} \in \mathcal{P}([0,1])$ with mean curve $m(t_n) \sim \frac{A}{e(n)}$ where we use the abbreviation

$$e(n) = e^{\alpha|t_n|}. \tag{3.45}$$

This is somewhat complicated by the possible interplay of probabilities and sizes in producing a given mean.

We begin by considering two special cases that illustrate the main idea. Let

$$\mu_{t_n} = (1 - \frac{p_1}{e(n)})\delta_{\frac{a_1}{e(n)}} + \frac{p_1}{e(n)}\delta_{a_2}, \tag{3.46}$$

so that $A = a_1 + p_1 a_2$. We can distinguish now different cases. The first case arises in our example if (small probability for an order one value a_2)

$$p_1 = 0, \ a_1 > 0, \tag{3.47}$$

in which case the mean curve satisfies

$$(3.48) \qquad e(n) \int x \mu_{t_n} dx \longrightarrow a_1,$$

the second case arises if (large probability for very small values)

$$(3.49) \qquad p_1 = 1, \ a_1 = 0, a_2 > 0,$$

in which case the mean-curve satisfies

$$(3.50) \qquad e(n) \int x \mu_{t_n} dx \longrightarrow \text{const}$$

given in (3.61).

We shall now consider these two cases each with a distinct flavour which then allows us, since they catch the key features, to see how we can treat the *general* case, which we call case 3 and for which we need to use an additional trick.

A key tool in all cases is the duality relation and the fact that the dual process consists of a collection of birth and death processes acting independently so that with generating functions we can analyse the dual expectation.

Case 1

Here we first treat the deterministic case $\mu_{t_n} = \delta_{\frac{a_1}{e(n)}}$. We will use the dual representation for the McKean-Vlasov limit together with (3.143) (giving below $W_n \Rightarrow W$) to compute the expected value of x_2 mass at time t, i.e. the mean of $\mathcal{L}_t(2)$. Note that then we consider the time stretch $|t_n| + t$ from the time point t_n where we have information about the state.

Observe that we know that the dual process at time T is a collection of K_T many independent birth and death processes, which have evolved for a time T and which in the large time limit $T \to \infty$ satisfy the *stable age distribution*. Therefore the state of the empirical measure of the birth and death processes at times $T + t$ is given by the stable size distribution, which arises as

$$(3.51) \qquad \int \mathcal{U}(\infty, ds) \nu_s \quad ,$$

where ν_s is the law of the state of a single birth and death process at time s starting with one particle at the occupied site.

We now fix some time $t > t_n$ and suppress the t-dependence in the notation. We need the generating function

$$(3.52) \qquad G_n(z) = E[z^{\zeta(t-t_n,i)}]$$

of the state of the birth and death process at a randomly chosen site i. To calculate we introduce the generating function

$$(3.53) \qquad G(z) = E[z^{\zeta}], \qquad 0 < z \leq 1,$$

where ζ represents the number of dual factors at a dual site under the stable size distribution and $(\zeta(i), i \in \mathbb{N})$ is an i.i.d. collection of such variables. We know that

$$(3.54) \qquad G_n \longrightarrow G \quad \text{as } n \to \infty,$$

$$(3.55) \qquad G'_n \longrightarrow G' \quad \text{on the unit circle as } n \to \infty.$$

Then noting that $G'(1) = B = \frac{\alpha+\gamma}{c}$ and using (3.167):

$$(3.56) \qquad \int_0^1 (1-x)\widetilde{\mathcal{L}}_t^n(dx) = E\left[\prod_{i=1}^{K_{t-t_n}} (1-\frac{a_1}{e(n)})^{\zeta(t-t_n,i)}\right] = E\left[\prod_{i=1}^{K_{t-t_n}} G_n(1-\frac{a_1}{e(n)})\right]$$
$$= E\left[\left(G(1-\frac{a_1}{e(n)})\right)^{W_n e(n) e^{\alpha t}(1+o(1))}\right], \text{ as } n \to \infty$$
$$\underset{n\to\infty}{\longrightarrow} E[e^{-W a_1 B e^{\alpha t}}]$$
$$\sim 1 - E[W] a_1 B e^{\alpha t} + O(e^{2\alpha t}), \text{ as } t \to -\infty.$$

From the first to the second line we have approximated G_n by G and used at the same time the convergence to the stable age distribution. To go from the second to the third line we use that $W_n \longrightarrow W$ as $n \to \infty$.

Therefore as $t \to -\infty$:

$$(3.57) \qquad m(t) \sim [E[W] a_1 B] e^{\alpha t} + O(e^{2\alpha t}).$$

Case 2

In case 2 the number of dual sites occupied in the dual process which have the value a_2 at time t_n, is denoted $Z_n(t, a_2)$, and is $\mathrm{Bin}(W_n e(n) e^{\alpha t}, \frac{1}{e(n)})$ distributed. Hence we get

$$(3.58) \qquad \int_0^1 (1-x)\widetilde{\mathcal{L}}_t^n(dx) = E[\prod_{i=1,\dots,K_{t-t_n}} (1-a_2)^{\zeta(t-t_n,i)})].$$

The r.h.s. of (3.58) is approximated as above as in case 1 (see (3.56)) for $n \to \infty$ by

$$(3.59) \qquad E\left[(G(1-a_2))^{Z_n(t,a_2)}\right].$$

Then $Z_n(t, a_2)$ converges as $n \to \infty$ in distribution to a Poisson distribution with parameter $We^{\alpha t}$.

For the calculation it is convenient to *condition on* W and read expectations as expectations for given W and write $\widetilde{E}$ for this conditional expectation suppressing the condition in the notation. We conclude that as $n \to \infty$ and then $t \to -\infty$:

$$(3.60) \qquad \widetilde{E}[\int_0^1 (1-x)\widetilde{\mathcal{L}}_t^n(dx)] \longrightarrow e^{[G(1-a_2)-1]We^{\alpha t}}, \text{ as } n \to \infty$$
$$\sim 1 - [(1-G(1-a_2))] W e^{\alpha t} + O(e^{2\alpha t}), \text{ as } t \to -\infty.$$

Hence as $n \to \infty$ and then for $t \to -\infty$ we get:

$$(3.61) \qquad E[\int_0^1 x\widetilde{\mathcal{L}}_t^n(dx)] \sim [(1-G(1-a_2))] E(W)] e^{\alpha t} + O(e^{2\alpha t}).$$

In other words also in this case $m(t) = Const \cdot e^{\alpha t} + O(e^{2\alpha t})$.

Case 3

We now turn to the general case. The key idea is to focus on the components where we do have type-2 mass even as $t_n \to -\infty$, which is achieved using the Palm measure. Consider the family $\{\widehat{\mu}_{t_n}\}_{n\in\mathbb{N}}$ of Palm measures on $[0,1]$:

$$(3.62) \qquad \widehat{\mu}_{t_n} := \frac{x\mu_{t_n}(dx)}{\int_0^1 x\mu_{t_n}(dx)}$$

and denote the normalizing constant again as in case 1,2 by

$$e(n) = \left(\int_0^1 x\mu_{t_n}(dx)\right)^{-1}. \tag{3.63}$$

Since the family $\{\widehat{\mu}_{t_n}\}_{n\in\mathbb{N}}$ is tight (by construction the measures are supported on $[0,1]$) we can find a subsequence and this sequence we denote again by $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to -\infty$ and $\widehat{\mu}_{t_n}$ converges. This means we can find a measure $\widehat{\mu}$ on $[0,1]$ (with $\widehat{\mu}(\{0\}) = 0$) such that

$$\frac{x\mu_{t_n}(dx)}{\int_0^1 x\mu_{t_n}(dx)} \Longrightarrow a\delta_0 + \widehat{\mu}, \quad \text{as } n \to \infty. \tag{3.64}$$

Moreover there exists a scale s_n, with $s_n > 0$ and $s_n \to 0$ such that

$$\frac{\int_0^{s_n} x\mu_{t_n}(dx)}{\int_0^1 x\mu_{t_n}(dx)} \to a. \tag{3.65}$$

We shall study the contributions due to the part $a\delta_0$ and due to $\widehat{\mu}$ first separately in (i) and (ii) below and then join things to get the total effect in (iii).

(i) We begin with the effect from $a\delta_0$. For this purpose consider the measures $(\nu_n)_{n\in\mathbb{N}}$ on $[0,1]$ defined by setting for every interval (c,d), with $0 \le c < d \le 1$:

$$\nu_n((c,d)) = \frac{\int_{ds_n}^{cs_n} x\mu_{t_n}(dx)}{\int_0^{s_n} x\mu_{t_n}(dx)}. \tag{3.66}$$

This allows us to consider the contribution due to the terms leading to $a\delta_0$ in the expectation in the duality relation. Define

$$(\widetilde{\nu}_n)_{n\in\mathbb{N}} \tag{3.67}$$

as the empirical measure from a sample of size $e^{W_n\alpha(t-t_n)}$ sampled from ν_n. Now we consider the state of the dual process at time $t - t_n$ and acting on the state at time t_n but considering only the contributions from sites with values in the interval s_n, which means that we can represent the state at time by an empirical measure as follows. As $n \to \infty$ we have:

$$\int_0^1 (1-x)\widetilde{\mathcal{L}}_t^n(dx) \sim E\left[\exp\left(W_n e^{\alpha t}\int_0^1 \log G(1-ys_n)\widetilde{\nu}_n(dy)\right)\right]. \tag{3.68}$$

By (3.126) $G''(1) < \infty$ and therefore the error term in (3.68) is $O(s_n)$. Then taking the limit as $n \to \infty$ we get using relation (3.65) that:

$$\begin{aligned} &E\left[\exp\left(W_n e^{\alpha t}\int_0^1 \log G(1-ys_n)\widetilde{\nu}_n(dy)\right)\right] \\ &\to E\left[\exp\left(-G'(1)We^{\alpha t}a\right)\right] \qquad \text{as } n \to \infty \\ &\sim 1 - G'(1)E[W]e^{\alpha t}a \qquad \text{as} \quad t \to -\infty. \end{aligned} \tag{3.69}$$

Hence in this case we get if there are only contributions from $a\delta_0$, i.e. $\widehat{\mu} \equiv 0$, that:

$$m(t) = (G'(1)E[W]a)e^{\alpha t} + O(e^{2\alpha t}), \tag{3.70}$$

i.e. $A = G'(1)E[W]a$.

(ii) Next we consider the contribution if they are only coming from $\widehat{\mu}$. We have to "return" here from the Palm $\widehat{\mu}$ to a measure μ which need not be a probability measure since x^{-1} diverges at $x = 0$. Note that for any $\varepsilon > 0$ the set of finite measures given by

$$\{e(n)(1_{x>\varepsilon})\mu_{t_n}(dx)\}_{n\in\mathbb{N}}, \tag{3.71}$$

is tight in the space of finite measures on $\mathcal{M}([0,1])$ and we can choose a convergent subsequence. We can then obtain a convergent sequence, we call again $(t_n)_{n\in\mathbb{N}}$ such that:

$$e(n)(1_{x>\varepsilon})\mu_{t_n}(dx) \Rightarrow \mu, \text{ as } n\to\infty \tag{3.72}$$

where μ is a measure on $(0,1]$ with

$$\int_0^1 x\mu(dx)<\infty, \text{ i.e. in particular } \mu([\varepsilon,1])<\infty \text{ for all } \varepsilon>0. \tag{3.73}$$

We denote

$$\mu_\varepsilon = \text{ restriction of } \mu \text{ to } [\varepsilon,1]. \tag{3.74}$$

By the dual representation we then have under the assumption

$$\mu_\varepsilon \text{ is discrete with atoms at } x_1,x_2,\cdots,x_M, \tag{3.75}$$

that with $x_{g_n(i)}$ being the value at site i at time t_n the representation:

$$\int_0^1 (1-x)\widetilde{\mathcal{L}}^n(dx) = E[\prod_{i=1,\dots,K_{t-t_n}} (1-x_{g_n(i)})^{\zeta(t-t_n,i)}]. \tag{3.76}$$

Denote by $Z_n(t,x_j)$ the number of dual sites occupied in the dual process which have at time t_n the value x_j. The r.h.s. of (3.76) we approximate again for $n\to\infty$ by

$$E\left[\prod_j G(1-x_j)^{Z_n(t,x_j)}\right]. \tag{3.77}$$

We know

$$\mathcal{L}[(Z_n(t,x_j))_{j=1,\cdots,M}] = Mult\left(W_n e(n)e^{\alpha t}, (\frac{\mu(x_j)}{e(n)})_{j=1,\cdots,M}\right). \tag{3.78}$$

The assumption (3.75) is removed as follows. For a general μ_ε, we consider the random measure

$$Z_n(t,dx) \text{ (the number of dual sites at which } x_2\in dx) \tag{3.79}$$

which converges as $n\to\infty$ in distribution to an *inhomogeneous Poisson random measure* with the intensity measure

$$\mu(dx)We^{\alpha t}. \tag{3.80}$$

For the calculation it is convenient to condition on W and read expectations as expectations for given W and write $\widetilde{E}$ for this object. We conclude that as $n\to\infty$ and then $t\to-\infty$ with the same reasoning as in the other cases:

$$\begin{aligned}\widetilde{E}[\int_0^1 (1-x)\widetilde{\mathcal{L}}^n_t(dx)] &\to e^{\int_0^1 (G(1-x)-1)\mu(dx)We^{\alpha t}} &&, \text{as } n\to\infty \\ &\sim 1-[\int (1-G(1-x))\mu(dx)]We^{\alpha t}+O(e^{2\alpha t}) &&, \text{as } t\to-\infty.\end{aligned} \tag{3.81}$$

Hence taking first $n\to\infty$ we then get for $t\to-\infty$ the expansion:

$$E[\int_0^1 x\widetilde{\mathcal{L}}^n_t(dx)] \sim [\int_0^1 (1-G(1-x))\mu(dx)]E[W]e^{\alpha t}+O(e^{2\alpha t}), \tag{3.82}$$

which then implies that the mean curve satisfies if $a = 0$, that

$$m(t) = Ae^{\alpha t} + O(e^{2\alpha t}), \tag{3.83}$$

with

$$A = E[W] \int_0^1 (1 - G(1-x))\mu(dx). \tag{3.84}$$

(iii) Finally we have to use the results in the two different cases where $\widehat{\mu} \equiv 0$ respectively $a = 0$, to get the answer in full generality. Combining the expressions for the two contributions we obtain as $n \to \infty$ and then considering the expansion as $t \to -\infty$ that

$$\int_0^1 (1-x)\widetilde{\mathcal{L}}_t^n(dx) \underset{n\to\infty}{\longrightarrow} E\left[e^{\int (G(1-x)-1)\mu(dx) W e^{\alpha t}} e^{-WG'(1)e^{\alpha t}a}\right] + O(e^{2\alpha t}), \tag{3.85}$$

and as a consequence for $t \to -\infty$ the r.h.s. is asymptotically given by

$$\sim \left([\int_0^1 [G(1-x) - 1]\mu(dx) + aG'(1)]E[W] \right) e^{\alpha t} + O(e^{2\alpha t}) = A \cdot e^{\alpha t} + O(e^{2\alpha t}) \tag{3.86}$$

with

$$A = \left(\int_0^1 [G(1-x) - 1]\mu(dx) + aG'(1) \right) E[W]. \tag{3.87}$$

Finally we have to exclude the possibility that the choice of different subsequences in (3.40), (3.64), (3.72) (and hence of different values for A) would give a different mean curve. However this would lead to a contradiction since this would give two values to the same quantities. Therefore the resulting mean curve is independent of the choice of subsequence $(t_n)_{n\in\mathbb{N}}$ of starting times.

This means that we have proved that an entrance law with at most exponentially growing mean has the property that the mean curve has asymptotically as $t \to -\infty$ the form

$$m(t) = A \cdot e^{\alpha t} + O(e^{2\alpha t}), \tag{3.88}$$

with A given by the formula (3.87) which completes the first of the two points we specified around (3.44).

We next need to argue that in fact such an entrance law exists by showing that actually $\widetilde{\mathcal{L}}^n$ (from (3.42)) converges along a subsequence. Due to the continuity of $\mathcal{L}_t$ in the initial state at time t_0, the tightness is straightforward. We have to show that first of all the limit is a solution and secondly the mean curve has the desired asymptotics. To show that we obtain in the limit a solution we argue as in the proof of part (c) of the proposition. It remains to verify the asymptotics of the mean curve.

For this purpose we use the same line of argument as above for the first moment of $x_2(t)$ to show that also all higher moments of $x_2(t)$ converge as $n \to \infty$ for every fixed t, so that we have convergence of the processes using the Feller property. For that we observe that we then simply have to start the dual process with k-initial particles and then we carry out the same calculations, since the CMJ-theory works also for the CMJ-process started with a different internal state of the starting site. We omit the straightforward modification.

Step 2 *Uniqueness of solution with the $t \to -\infty$ asymptotics*

We next show that if the mean curves of two entrance laws differ at most by $o(e^{\alpha t})$ as $t \to -\infty$, then they are identical. This then allows to conclude that if we prescribe the value A then there is exactly one entrance law with this asymptotics as $t \to -\infty$.

For that purpose we consider two solutions $(\mathcal{L}_t^1)_{t\in\mathbb{R}}, (\mathcal{L}_t^2)_{t\in\mathbb{R}}$ such that the mean curves $(m^\ell(t))_{t\in\mathbb{R}}$ satisfy for $\ell = 1, 2$ that $m^\ell(t) \sim A\exp(\alpha t)$ as $t \to -\infty$. vbbbbbb We consider a sequence $(t_n)_{m\in\mathbb{N}}$ with $t_n \to -\infty$ as $n \to \infty$ and we fix a time point $t \in \mathbb{R}$. Then we define the two processes we apply the duality relation at time t

$$(3.89) \qquad (\widetilde{\mathcal{L}}_s^{n,1})_{s\geq t_n} \quad , \quad (\widetilde{\mathcal{L}}_s^{n,2})_{s\geq t_n},$$

which have at time t_n the initial distributions given by

$$(3.90) \qquad \mathcal{L}_{t_n}^1 \text{ resp. } \mathcal{L}_{t_n}^2.$$

We apply the duality relation at time t. Our goal is to show that the mean curve satisfies

$$(3.91) \qquad m^1(t) = m^2(t), \quad \forall\, t \in \mathbb{R},$$

by proving that for every $t \in \mathbb{R}$

$$(3.92) \qquad \widetilde{m}^{n,1}(t) = \widetilde{m}^{n,2}(t) + o(1) \text{ as } n \to \infty.$$

We first verify this in Case 1. Let $\widetilde{\mathcal{L}}^{n,1}$, $\widetilde{\mathcal{L}}^{n,2}$ be two solutions as in case 1 but such that $m^1(t_n) - m^2(t_n) = o(\frac{1}{e(n)})$. Then by (3.56),

$$(3.93) \qquad \begin{aligned} &\left|\int_0^1 (1-x)[\widetilde{\mathcal{L}}_t^{n,1}(dx) - \widetilde{\mathcal{L}}_t^{n,2}(dx)]\right| \\ &\leq E\left[\left|\prod_{i=1}^{K_{t-t_n}} (1 - \frac{a_1}{e(n)} + o(\frac{1}{e(n)}))^{\zeta(t-t_n,i)} - \prod_{i=1}^{K_{t-t_n}} (1 - \frac{a_1}{e(n)})^{\zeta(t-t_n,i)}\right|\right] \\ &= E\left[\left|\prod_{i=1}^{K_{t-t_n}} G_n(1 - \frac{a_1}{e(n)} + o(\frac{1}{e(n)})) - \prod_{i=1}^{K_{t-t_n}} G_n(1 - \frac{a_1}{e(n)})\right|\right] \\ &\leq E[G'(1)W_n e(n) e^{\alpha t} o(\frac{1}{e(n)})] \to 0 \text{ as } n \to \infty, \end{aligned}$$

where we have first used that we can split the expectation in the duality relation into one over the initial state and then over the dual dynamic, furthermore for $z \in [0,1]$, $G_n'(z) \leq G_n'(1) \leq G'(1)$ (the latter by coupling). Hence we have $m^1 = m^2$ in this case.

In case 2 there iis $o\left(\frac{1}{e(n)}\right)$ change in the mean curve resulting from the change of a_2 to $a_2 + o\left(\frac{1}{e(n)}\right)$. Moreover given a change of the frequency $\frac{p_1}{e(n)}$ to $\frac{p_1}{e(n)} + o\left(\frac{1}{e(n)}\right)$ we have $Z_n(t, a_2)$ is $\text{Bin}\Big(W_n e(n) e^{\alpha t}, \frac{1}{e(n)} + o\left(\frac{1}{e(n)}\right)\Big)$ and the limiting Poisson distribution is not changed by the $o\left(\frac{1}{e(n)}\right)$ perturbation.

Again, the general case follows by a combination of the arguments of cases 1 and 2 exactly as in case 3 in Step 1.

Step 3 *Uniqueness given the mean curve*

To complete the proof for uniqueness of the entrance law with specified behaviour as $t \to -\infty$, it remains to show that for a given mean curve $(m(t))_{t\in\mathbb{R}}$ there is a unique solution $(\mathcal{L}_t^*)_{t\in\mathbb{R}}$ to the McKean-Vlasov equation having this mean curve. Since the McKean-Vlasov dynamics is unique for $t \geq t_0$ it suffices to prove uniqueness of $\mathcal{L}_{t_0}^*$ for arbitrary t_0.

Now consider the tagged component in the McKean-Vlasov system $(x_2^\infty(1,t))_{t\geq t_0}$ assuming that the mean curve $(m(t))_{t\in\mathbb{R}}$ satisfies

$$(3.94) \qquad \limsup_{t\to-\infty} m(t) = 0.$$

Then $\mathcal{L}_t^*$ is the law of a tagged component conditioned on the realization of the mean curve and this conditioned law is a weak solution of the wellknown SDE

$$dx(t) = c(m(t) - x(t))dt + sx(t)(1 - x(t))dt + \sqrt{x(t)(1 - x(t)}dw(t), \tag{3.95}$$

(cf. (1.21)) which has a unique weak solution given an *initial* value at time t_0. However we must prove the existence and uniqueness of a solution with time running in $\mathbb{R}$ and satisfying

$$\lim_{t \to -\infty} x_2^\infty(1, t) = 0. \tag{3.96}$$

We construct a "minimal solution" $\tilde{x}_2^\infty(1, t)$ for this problem by starting with a sequence of solutions $x_{(n)}(t)$ of (3.95) (driven by independent Brownian motions $w_n(t)$) starting with value $x^n(t_n) = 0$ at times $(t_n)_{n\in\mathbb{N}}$ with $t_n \to -\infty$. We then construct a coupled version $\{\widehat{x}^m : m \geq m_0\}$ by coupling paths x^m and x^n, $n \leq m$ at a time where they collide. It can be verified that the coupled system $\{(\widehat{x}^n(t))_{t\in\mathbb{R},n\in\mathbb{N}}\}$ is a stochastically monotone increasing sequence as $n \to \infty$ with limit $\widehat{x}_2^\infty(1, t)$. If we assume *existence* of a solution of (3.95) with time index $\mathbb{R}$ and mean curve $m(t)$ we then know it will be stochastically bounded below by $\widehat{x}_2^\infty$.

Next we need uniqueness in law of the solution and we shall show that it must agree with $\widehat{x}_2^\infty$, which then automatically must also be a solution. We observe that any solution to (1.106) must have zeros at $-|t|$ for arbitrarily large $|t|$ (cf. classical property of Wright-Fisher diffusions [S1]). If it has a zero at $t^* > t_n$, then we can couple it with $\widehat{x}_n$ and therefore we can couple the diffusion with the minimal one therefore the considered diffusion must agree with a version of $\widehat{x}_2^\infty$ after the coupling time. Combined with the uniqueness property of the McKean-Vlasov equation this also proves uniqueness of $\mathcal{L}_t^*$, $t \geq t_0$. Since t_0 is arbitrary, we obtain the *uniqueness* for all $t \in \mathbb{R}$. This completes the proof of (e) of the Proposition 1.3.

(f) Finally, we consider a *random* solution $\{\mathcal{L}_t : t \in \mathbb{R}\}$ to the McKean-Vlasov equation. Since $\mathcal{L}_t$ is a.s. a solution and since by assumption it satisfies a.s. the growth condition with $A \in (0, \infty)$, it follows that it is given a.s. by a (random) time shift of the standard solution.

3.2.3 The structure of the dual process and a Crump-Mode-Jagers process

The key tool for the proof of the emergence results is the dual representation for the system of N exchangeably interacting sites, namely, the dual process $(\eta_t, \mathcal{F}_t^+)$ (note the dependence on N is suppressed in the notation).

Since in the case of two types, $\mathbb{K} = \{1, 2\}$ the two frequencies of type one and two add up to 1, it suffices to determine the law of $x_1(\frac{\log N}{\alpha} + t)$ and $\Xi_N(\frac{\log N}{\alpha}) + t, \{1\})$ in the limit $N \to \infty$. This will, as we shall see in Subsubsection 3.2.11, involve computing (ℓ, k)-moments, i.e. the product of k-th moments at ℓ-different sites for the N-site system and for the random McKean-Vlasov limit. The calculation is carried out using the dual process started with k particles at each of ℓ different sites. We denote the number of particles in the dual process at time t with such an initial condition by

$$\Pi_t^{N,k,\ell}, \text{ we often write } \Pi_t^N \text{ if } (k, \ell) \text{ are fixed.} \tag{3.97}$$

First, to prove that the mass of the inferior type at a tagged site goes to zero in time scale $\alpha^{-1} \log N + t$ as $N \to \infty$ and then $t \to \infty$, it suffices to work with $E[x_1(\cdot)]$. The calculation of this expectation involves only one initial particle for the dual process. To carry out the calculation we assume that $x_1(0, i) = 1$ for $i = 1, \cdots, N$ and take the initial element for the dual process

$$\eta_0 = \delta_{i_0}, \tag{3.98}$$

i.e. one particle at a tagged site in $i_0 \in \{1, \ldots, N\}$. Note that because of migration the law of this process depends indeed heavily on N if t is large enough.

The number of factors at each site form a system of dependent birth and death processes with immigration, i.e. a process with state space $(\mathbb{N}_0)^N$, denoted

(3.99) $$(\zeta^N(t))_{t\geq 0} \text{ with } \zeta^N(t) = \{\zeta^N(t,i), i = 1,\cdots,N\}.$$

In other words

(3.100) $$\Pi_t^{N,k,\ell} = \sum_{i=1}^N \zeta^N(t,i),$$

and $\zeta^N(0)$ has exactly ℓ-non zero components each containing exactly k factors.

Next we describe the dynamic of ζ^N. Consider first a single component i. If the current state of ζ_t^N at a site i is $x \in \mathbb{N}_0$, then

(3.101) the immigration rate of one new individual to this site is $c(N^{-1}(\Pi_t^{N,k,\ell} - x))$, where every individual at the other sites has the same chance to be selected for migration to i,

(3.102) the emigration rate for one individual at this site is $c\dfrac{N-1}{N}x$, the individual moves to a randomly chosen different site,

(3.103) the death rate, i.e. rate for the death of one individual at this site is $(d/2)x(x-1)$, and

(3.104) the birth rate for one new individual is sx.

The dependence of the birth and death processes $(\zeta(t,i))_{t\geq 0}$, $i = 1,\cdots,N$ at different sites arises from the migration transition in (3.101), namely the immigration is coupled with the emigration at another site.

Note that because of the exchangeability of X^N and the initial state $X^N(0)$ it suffices for the dual ζ^N to keep track of

- the number of occupied sites,
- the number of individuals, i.e. factors, at each occupied site.

We now introduce the notation needed to describe the dual population. First the process

(3.105) $$\{K_t^N\}_{t\geq 0}$$

that counts the number of *occupied sites* at time t. Secondly the process recording the *age and size* of each site (age is counted from the time of "birth", that is, the time when the site first became occupied the last time). Note that each site is uniquely identified by its birth time. Equivalently, due to exchangeability it suffices to keep track of the number of occupied sites K_t^N and the empirical age and size distribution at time t, a measure on $\mathbb{R}^+ \times \mathbb{N}$ denoted

(3.106) $$\Psi^N(t, ds, dy),$$

where $\Psi^N(t,(a,b],y)$ denotes the number of sites in which the birth time, that is, the time at which the initial immigrant arrives, lies in the time interval $(a,b]$ and the current size is $y \in \mathbb{N}$.

We will use the abbreviation

(3.107) $\Psi^N(t, ds) = \Psi^N(t, ds, \mathbb{N})$.

Consider next for a fixed $t \in \mathbb{R}^+$ the collection of processes

(3.108) $\{(\zeta_s^N(u))_{u \geq s},\ s \in \text{ supp } \Psi^N(t, \cdot, \mathbb{N})\}$,

which denotes the size of the population at a site having birthtime s, that is, one particle arrives at time s. In the absence of collisions these processes for different birth times are independent birth and death processes and their distributions depend only on their current age. However in the presence of *collisions* these are *coupled* (by migration) and no longer independent.

The number of occupied sites fluctuates due to migration to *new* sites respectively jumps from sites with one particle to other occupied sites. Note that the probability of a jump of the latter type is non-negligible only if the number of occupied sites is comparable to the total number of sites. Hence up to the time that the number of occupied sites reaches $O(N)$ the number of occupied sites is (asymptotically) non-decreasing and the birth and death processes at different sites are asymptotically independent.

We obtain an upper bound for the growth of $(K_t^N)_{t \geq 0}$ if we suppress all collisions and assume that always a new site is occupied. In order to construct this process, we enlarge the geographic space from$\{1, \cdots, N\}$ to $\mathbb{N}$ and drop in (3.101) all jumps to occupied sites replacing them by jumps to a new site, namely the free one with the smallest label. We refer to this process as the *McKean-Vlasov dual* since it arises by taking in our model of N exchangeable sites the limit as $N \to \infty$ of the dual process. This process is denoted (if we want to stress how it arises we add the superscript ∞):

(3.109) $(\zeta_t) = (\zeta_t(1), \zeta_t(2), \cdots)$

and the process of the total number of individual by

(3.110) $(\Pi_t^{(k,\ell)})_{t \geq 0}$,

and ζ has the following markovian dynamic.

If we have at a site $i \in \mathbb{N}$ occupied with $x \in \mathbb{N}$ individuals then we have

(3.111) one death at rate $\dfrac{d}{2} x(x-1)$,

(3.112) one birth at rate sx

and if j is the smallest index with $\zeta_t(j) = 0$ then

(3.113) at rate $c\zeta(i)$ an individual moves from i to j.

Since in the duality relation we only need the occupation numbers of occupied sites, we can *ignore migration steps of single factors* to another unoccupied site. Therefore in the migration rate in (3.113) we take

(3.114) $c\zeta_t(i)1_{(x \geq 2)}$ instead of $c\zeta_t(i)1_{(x \geq 1)}$.

This way we obtain, completely analogous to (3.105) - (3.108) new processes which we denote

(3.115) $(K_t)_{t \geq 0}, (\Psi(t, ds, dj))_{t \geq 0}, \{(\zeta_s(u))_{u \geq s}, s \in \mathbb{R}^+\}$.

Then we know that for every $t \in \mathbb{R}^+$ the collection

(3.116) $\{(\zeta_s(u))_{u \geq s}, s \in supp \Psi(t, \cdot, \mathbb{N})\}$

consists of *independent* birth and death processes.

We can show that

$$K_t^N \leq K_t \quad , \quad \text{stochastically for all } t \geq 0 \text{ and } N \in \mathbb{N}. \tag{3.117}$$

Namely on the r.h.s. we suppress collisions and at each jump occupies a *new* site. Since after a collision we can have coalescence with one of the particles already at that site, a coupling argument shows that we have fewer particles as in the model with collisions, in fact strictly fewer with positive probability.

Then the dual $(\eta_t, \mathcal{F}_t^+)$ on the N-site model evolves exhibiting the following features. The dual population consists of an increasing number, Π_t^N, of $(1_{\{1\}})$ factors (due to birth events related to selection). Each factor $(1_{\{1\}})$ (and therefore the product) can jump to 0 by rare mutation at rate $\frac{m}{N}$ (because the factor jumps to $(1_{\{2\}})$, which becomes 0 in the dual expression for $E[x_1^N(t)]$ by (2.10 with $n = 1$) since $x_2^N(0) = 0$). Therefore asymptotically as $N \to \infty$ to have non-zero probability of such a jump to zero to occur, we need $\Pi_t^N \sim O(N)$. This means that asymptotically the probability that a mutation event occurs becomes positive only for sufficiently large times such that the number of $(1_{\{1\}})$ factors in $\mathcal{F}_t^+$ reaches $O(N)$.

In order to see the meaning of this behaviour for the original process recall the *first moment duality relation* (assuming $x_1^N(0) = 1$) rewritten in terms of the quantities introduced (3.105)-(3.108):

$$\begin{aligned} E[x_1^N(t)] \quad &= E[\exp(-\frac{m}{N}\int_0^t \Pi_u^{N,1,1} du)] \\ &= E\left[\exp\left(-\frac{m}{N}\int_0^t \left(\int_0^u \zeta_v^N(u)\Psi^N(u,dv)\right) du\right)\right]. \end{aligned} \tag{3.118}$$

In order to prove Proposition 1.7,(1.66), we therefore need to show that asymptotically the probability that a mutation event occurs by time $T_N + t$ goes to 0 as $N \to \infty$ and $t \to -\infty$.

A complication in the analysis of the dual process arises in that if a migrant lands at a previously occupied site, that is, a *collision occurs*, then the different birth and death processes are not independent due to coalescence with its quadratic rate and we must take this into account when K_t^N reaches $O(N)$ since there is then a non-negligible probability of a collision.

We will therefore study in Subsubsections 3.2.4 and 3.2.5 first the collision-free regime of the dual dynamic and in Subsubsection 3.2.2 its consequences for the process and then in Subsubsections 3.2.6-3.2.10 the regime with collisions.

3.2.4 The dual in the collision-free regime: the exponential growth rate

Due to coalescence (if $d > 0$) the total number of $(1_{\{1\}})$ factors is locally stochastically bounded but due to migration the number of occupied sites can grow and with it the total number of factors. We need to find α such that the number of factors is at time $\alpha^{-1}\log N + t$ of order N for $t \geq t_0$ and N large and the integrated lifetime of all factors up to that time is $O(N)$ as $N \to \infty$, with smaller and smaller number of factors as $t \to -\infty$ and a diverging number as $t \to \infty$. We therefore will fix a candidate α and then show lower and upper bound on the number of factors. We will introduce this α as the exponential growth rates of the number factors as $N \to \infty$ as follows.

If $c > 0$ and $N = \infty$ (i.e. "mean-field migration"), then the number of factors (i.e. Π_t^∞) would grow exponentially fast as we prove below. We will now first work with this scenario of mean-field migration in Step 1 providing a candidate for α and an upper bound. In the next subsubsection we return to the effect of collisions, which can arise if $N < \infty$ and we have migration to a previously occupied site to get a lower bound.

In order to verify (1.66), first note that by Jensen and (3.118)

$$E[x_1^N(t)] \geq \exp\left(-\frac{m}{N}E\left[\int_0^t [\int_0^u \zeta_v^N(u)\Psi^N(u,dv)]du\right]\right). \tag{3.119}$$

In the next three steps we analyse first $\zeta_v^N(u)$ given K_τ^N for τ from 0 up to u to get an estimate for the growth rate of the integral in the r.h.s. in terms of the process K^N of (3.119) and then in the second step we focus on the growth of K_u^N in u and in third step will then be simple, we combine the results to get that the α we have chosen is an upper bound for the exponential growth rate (in t) of the double-integral in the r.h.s. of (3.119).

Step 1 *Sufficient condition for (1.66)*
We next show in order to control the r.h.s. of (3.119) that

$$\limsup_{N\to\infty} E[\zeta_v^N(u)|\sigma\{K_s^N : s \leq u\}] \leq L \tag{3.120}$$

and L is non-random, i.e. a constant. Given α and recalling (3.119) in order to establish the upper bound on the emergence time, it suffices once we have (3.120) to show that:

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{N}\int_0^{\frac{\log N}{\alpha}-t} E[K_s^N]du = 0, \tag{3.121}$$

which we will do in the Step 2.

To prove (3.120) consider the collection of auxiliary processes (here $\iota \in \mathbb{R}^+$)

$$(\zeta_{0,t}(\iota))_{t\geq 0}, \tag{3.122}$$

which as it turns out bounds our process from above in a site colonized at time 0 at time t for proper choice of the parameter ι. It has the dynamic of a birth and death process with *linear birth rate sk, quadratic death rate* $\frac{d}{2}k(k-1)$, *emigration* at rate $ck1_{(k\geq 2)}$ and *immigration* of a particle at rate ι. The parameter ι we choose later. The case $\iota = 0$ corresponds to the McKlean-Vlasov dual, i.e. the dual in the collision-free regime. The process has a unique equilibrium state

$$\zeta_{0,\infty}(\iota), \tag{3.123}$$

and is ergodic (with exponential convergence to equilibrium).

We have (using a standard coupling argument) in stochastic order:

$$\zeta_{0,\infty}(0) \leq \zeta_{0,\infty}(\iota). \tag{3.124}$$

A standard coupling argument can then be used to show that for all $t \geq 0$ and every s the random variable (recall we start in one particle)

$$\zeta_{s,t}(\iota) \text{ is stochastically dominated by } \zeta_{0,\infty}(\iota). \tag{3.125}$$

Moreover writing down the differential equation for the second moment of $\zeta_{0,t}$ it is easy to verify that for every $\iota \geq 0$:

$$E[(\zeta_{0,\infty}(\iota))^2] < \infty. \tag{3.126}$$

With our McKean-Vlasov dual process we can associate a mean-field birth and death process where the parameter ι is time-dependent, namely the immigration ι_t is the *current mean* of the process. This process arises as the limit $N \to \infty$ of the process with migration between N sites

according to the uniform distribution and starting with a positive intensity of particles. The mean-field birth and death process has a unique equilibrium which we reach if we start with one particle. Namely it can be verified that the fixed point equation

$$(3.127) \quad \iota_* = E[\zeta_{0,\infty}(\iota_*)]$$

has a finite solution which gives the expectation for the equilibrium of the McKean-Vlasov dual and that in fact determines the McKean-Vlasov equilibrium.

To show this we consider the nonlinear Kolmogorov equations for the process $(\zeta_{0,t}(\iota))_{t\geq 0}$ which are spelled out in all detail later on in (3.268) where we set in (3.268) $\alpha(t)u(t)\equiv\iota$. More precisely let $p_{k,\ell}(t)=p_{k,\ell}(\iota,t)$ denote the transition probabilities and set

$$(3.128) \quad m_t(\iota)=\sum_{j=2}^{\infty} jp_{1,j}(\iota,t),$$

$$(3.129) \quad m(\iota)=\lim_{t\to\infty}\sum_{j=2}^{\infty} jp_{1,j}(\iota,t).$$

To look for equilibria we look for fixed points $m(\iota)=\iota$. Using coupling we can show that $m(\iota)$ is monotone increasing, continuous, ultimately sublinear in ι and $\lim_{\iota\to 0} m(\iota)>0$ since the smallest state is 1 if we start in 1. Therefore there exists a unique

$$(3.130) \quad \text{largest solution } \iota^* \text{ of the equation } m(\iota)=\iota \text{ with } m_t\leq m(\iota^*).$$

If we insert this ι^* as parameter the mean remains constant and the equilibrium of the McKean-Vlasov dual process, is the unique equilibrium of the Markov process $(\zeta_{0,t}(\iota^*))_{t\geq 0}$.

We see again by coupling that for $\iota\leq\iota^*$:

$$(3.131) \quad \zeta_{0,\infty}(0)\leq\zeta_{0,\infty}(\iota)\leq\zeta_{0,\infty}(\iota^*).$$

We now show that this provides the bound required in (3.120). Note that at a tagged site in the exchangeable system of N sites the number of particles is given by

$$(3.132) \quad \zeta^N_{0,t}=\zeta_{0,t}((\iota^N_u)_{u\leq t}) \text{ with } \iota^N_u=\frac{\Pi^N_u}{N}\leq\frac{\Pi_u}{N},$$

where Π_t is the number of particles in the McKean-Vlasov dual process (without collisions), which can be constructed on the same probability space as Π^N, such that the inequality on the r.h.s. holds and such that

$$(3.133) \quad \zeta^N_{0,t}\leq\zeta_{0,t}((\frac{\Pi_u}{N})_{u\leq t})\leq\zeta_{0,\infty}(\iota^*).$$

A standard mean-field limit argument then shows that $(\zeta^N_{0,t},\Pi^N_t)$ converges in law as $N\to\infty$ to $(\zeta_{0,t}(\{m_u\}_{u\leq t});K_t)$, where in the latter the two components are independent.

Therefore using (3.131) and the construction of ι^* we see that:

$$(3.134) \quad \lim_{N\to\infty} E[\zeta^N_{0,t}|\Pi^N_s : s\leq t]=E[\zeta_{0,t}(\{m_u\}_{u\leq t})]\leq E[\zeta_{0,\infty}(\iota_*)]=\iota^*$$

and this proves (3.120) .

Step 2 *A Crump-Mode-Jagers process gives a Malthusian parameter α.*

To show (3.121) and to obtain a candidate for α we can use the collision-free mean-field dual Π_t in particular its ingredients K_t and ζ_t.

The process K in (3.115) has some important structural properties. Recall the process counts the number of occupied sites, each of which has an internal state, which consists of the number of particles at these sites. Each of the particles exceeding the first particle can migrate and produce a new occupied site. Whenever a new site is colonized then an independent copy of the basic one site process starts and evolves independently of the rest.

To analyse this object we recall the concept and some basic results on *supercritical Crump-Mode-Jagers branching processes* and in particular its *Malthusian parameter* α. Such a process is defined by the following properties.

The process counts the number of individuals in a branching population whose dynamics is as follows. Individuals can die or give birth to new individuals based on the following ingredients:

- individuals have a lifetime (possibly infinite),
- for each individual an independent realization of a point process $\xi(t)$ starting at the birth time specifying the times at which the individual gives birth to new individuals,
- different individuals act independently,
- the process of birth times is not concentrated on a lattice.

Let $(K_t)_{t \geq 0}$ be a process with the described structure. The corresponding *Malthusian parameter*, $\alpha > 0$ is obtained as the unique solution of

$$\int_0^\infty e^{-\alpha t} \mu(dt) = 1 \text{ where } \mu([0,t]) = E[\xi([0,t])], \tag{3.135}$$

with

$$\xi(t) \text{ counting the number of births of a single individual up to time } t, \tag{3.136}$$

(see for example [J92], [N] equation (1.4)).

In our case we have

$$\mu([0,t]) = c \int_0^t E[\zeta_0(s) 1_{(\zeta_0(s) \geq 2)}] ds. \tag{3.137}$$

Remark 17 *Note that in our model particles at singly occupied sites do not migrate. We can suppress the emigration step if only one particle is left, since in that case everything would simply start afresh at some other site. We have done this so that in the pre-collision regime the number of occupied sites equals the number of migrating particles. In this case the rate of creation of new sites due to migrations from a given site is given by $\zeta_0(s) 1_{\zeta_0(s) \geq 2}$ and the death rate when $\zeta_0(s) = 1$ is 0.*

However we can also consider the original model in which particles at singly occupied sites do migrate. Then sites have finite lifetimes *and the rate of production of new sites by a fixed site at time s is given by $\widetilde{\zeta}_0(s)$ where now $\widetilde{\zeta}_0(s)$ has death rate c when $\widetilde{\zeta}_0(s) = 1$. Then we have instead of (3.137) the equation*

$$\mu([0,t]) = c \int_0^t E[\widetilde{\zeta}_0(s)] ds. \tag{3.138}$$

However the Malthusian parameter describing the exponential growth of the number of sites occupied by at least one particle is the same in both cases as is shown by explicit calculation.

We therefore define α by (recall for ζ_0 (3.115) and above):

$$(3.139)\quad c\int\limits_0^\infty e^{-\alpha s} E[\zeta_0(s) 1_{(\zeta_0(s)\geq 2)}]ds = 1.$$

The first thing we want to know is that in our case

$$(3.140)\quad \alpha \in (0,s) \quad \text{if} \quad d > 0 \text{ and } \alpha \in (0,s] \text{ if } d \geq 0.$$

In the case of our dual process $K_t \leq N_t$ where N_t is a rate s pure birth process and therefore $\alpha \leq s$.

To show that the Malthusian parameter satisfies:

$$(3.141)\quad \alpha > 0$$

note first that when $d > 0$ still no deaths (in the number of occupied sites) occur and that a lower bound to the growth process is obtained by considering only those migrants which come from the birth of a particle at sites containing sofar only one particle and which then migrates before coalescence or the next birth occurs. This process of the number of such special sites is a classical birth and death process with in state k birth rate $s(\frac{c}{c+s+d})k$ and death rate zero which clearly has a positive Malthusian parameter (if $c > 0$) and hence (3.141) holds.

Next to see that if $d > 0$ then $\alpha < s$, note that a non-zero proportion of the new particles generated by a pure birth process at rate s dies due to coalescence and therefore $\alpha < s$ in this case.

If we know the Malthusian parameter, we need to know that it is actually equal to the almost sure growth rate of the population. It is known for a CMJ-process $(K_t)_{t\geq 0}$ that (Proposition 1.1 and Theorem 5.4 in [N]) the following basic growth theorem holds. If

$$(3.142)\quad E[X \log(X \vee 1)] < \infty,$$

where here $X = \int\limits_0^\infty e^{-\alpha t} d\xi(t)$, then

$$(3.143)\quad \lim_{t\to\infty} \frac{K_t}{e^{\alpha t}} = W, \text{ a.s. and in } L_1,$$

where W is a random variable which has two important properties, namely

$$(3.144)\quad W > 0,\ a.s. \text{ and } E[W] < \infty.$$

We now have to check that (3.142) holds in our case.

We know using (3.139) that $\int\limits_0^\infty u e^{-\alpha u}\mu(du) < \infty$, since we can bound the birth rate by the one in which the number of particles at each site is given by the equilibrium state $\mathcal{L}(\zeta_\infty)$, which has a finite mean. This also implies that assumption (3.142) holds. Namely, in equilibrium we can estimate

$$(3.145)\quad E[X^2] \leq \int\limits_0^\infty \int\limits_0^\infty e^{-\alpha(s+t)} E[d\xi(s) d\xi(t)] < \infty,$$

by using Cauchy-Schwartz and using (3.126).

Remark 18 *In the above we considered the dual $\mathcal{F}^+$ starting with one factor (particle) at one initial site. We can also consider the case with k initial particles at one site. In this case the same exponential growth occurs since (3.137) still holds but the corresponding random growth factor $W^{(k)}$ in (3.143) now depends on k. Similarly we can start with k particles at each of ℓ different sites and again get exponential growth with the same Malthusian parameter α but now with random variable $W^{(k,\ell)}$. This will be frequently used in the sequel of this section.*

Step 3 *Completion of argument*

To complete the argument we note that (3.121) follows from(3.143) (which implies that $K_t = e^{-\alpha t} W_t$ with $W_t \to W$) and (3.117) since

$$E(\int_0^{\alpha^{-1}\log N - t} K_s^N ds) \leq Const \cdot \frac{1}{\alpha} E(W) e^{\log N - \alpha t}. \tag{3.146}$$

We saw earlier that (3.121) is sufficient for (1.66) and the proof is complete that emergence does not happen before time $\alpha^{-1} \log N$ as $N \to \infty$.

3.2.5 The dual in the collision-free regime: Further properties

We shall need for the analysis of the regime with collisions some further properties concerning the dual in the collision-free regime. First, we obtain further properties of the growth constant W, more precisely its moments, and then the stable age distribution which among other things allows to give an explicit expression for the growth rate α. The random variable W is defined in (3.143) in terms of the process $(K_t)_{t \geq 0}$ introduced in (3.115), that is, we consider the dual process without collision.

Step 1 *Higher moments of W.*

In the sequel we will perform moment calculations which use:

Lemma 3.4 *(Higher Moments of W)*

For all n,

$$E[W^n] < \infty. \qquad \square \tag{3.147}$$

Proof of Lemma 3.4 By [BD], Theorem 1, W has a finite nth moment if and only if

$$E\left[\left(\int_0^\infty e^{-\alpha s} d\xi(s)\right)^n\right] < \infty, \tag{3.148}$$

where $\xi(t)$ denotes the number of particles that have emigrated by time t from a fixed site starting with 1 particle. We will verify this condition for the second moment.

$$\begin{aligned} E[\left[\int_0^\infty e^{-\alpha s} d\xi(s)\right]^2 &= 2E\left[\int_0^\infty \int_s^\infty e^{-\alpha s} e^{-\alpha(s+s'))} d\xi(s') d\xi(s)\right] \\ &= c^2 E\left[\int_0^\infty \int_s^\infty e^{-\alpha(s+s')} \zeta_0(s) E[\zeta_0(s')|\zeta_0(s)] ds' ds\right] \\ &= c^2 E\left[\int_0^\infty \int_s^\infty e^{-\alpha s} \zeta_0(s)\zeta_0(s') ds' ds\right]. \end{aligned} \tag{3.149}$$

Recall that here ζ_0 is a birth and death process with birth rate sk and death rate $ck1_{k\geq 2} + \frac{1}{2}dk(k-1)$. But then by explicit calculation $\sup\limits_s E(\zeta_0(s)^2) < \infty$, and therefore

$$2E\left[\int_0^\infty \int_s^\infty e^{-\alpha s} \zeta_0(s)\zeta_0(s') ds' ds\right] < \infty. \tag{3.150}$$

This completes the proof that $E(W^2) < \infty$.

Recalling that $\zeta_0(t)$ is for all t stochastically dominated by the equilibrium distribution $(p_k)_{k\in\mathbb{N}}$ of this birth and death process and that the latter satisfies for some $C < \infty$,

$$(3.151)\qquad p_k \le \frac{C^k}{k!}, \qquad \text{for all } k = 1, 2, \cdots$$

it follows that for every $n \in \mathbb{N}$,

$$(3.152)\qquad \sup_s E[(\zeta_0(s))^n] < \infty.$$

Then a similar argument to the second moment case argument given above verifies that nth moments of W are finite.

Remark 19 *Alternatively to the above we can argue as follows. In the pre-collision regime we can also represent the particle process as a branching Markov chain, more specifically a* branching birth and death process *(linear birth rate and quadratic death rate), that is a counting measure-valued process, on* $\mathbb{N}$*,*

$$(3.153)\qquad \Lambda_t \in \mathcal{M}(\mathbb{N}),$$

where $\Lambda_t(k)$ counts the number of sites occupied by exactly k partition elements. In other words the individuals of this new process Λ correspond to the occupied sites of the dual particle system and the location of individuals is now the size of the population at the sites of the dual particle system.

The dynamics of $(\Lambda_t)_{t\ge 0}$ are therefore given by

- *The branching. Individuals branch independently with a location-dependent branching rate, namely, an individual at k dies at rate ck and produces two new individuals with new locations, namely one individual at $k-1$ and a second one at 1. Since sites of size 0 that would be produced if a site of size 1 dies would not contribute to the production of new sites, they can be ignored. For this reason we suppress these deaths and set the death rate at k to be $ck1_{k\neq 1}$.*
- *The motion: the motion between locations is given by a Markov process $(\widetilde{\zeta}(t))_{t\ge 0}$ on $\mathbb{N}\cup\{\infty\}$, namely, a birth and death process $(\zeta(t))_{t\ge 0}$, where the birth rate is sk and the death rate $dk(k-1)/2$.*

In other words the CMJ process can be embedded in the branching birth and death process and the number of individuals in that process is the same as the number of occupied sites in our original CMJ-process.

We work with the probability generating function, M_t, of branching birth and death process Λ_t and use this to compute the distribution of the number of occupied sites as follows.

Define the generating function:

$$(3.154)\qquad M_t(k,z) = E_{\delta_k}[z^{\Lambda_t(1_{\{\mathbb{N}\}})}] = \sum_{j=1}^{\infty} p(t,k,j) z^j, \quad 0 < z \le 1,$$

where we write $\Lambda_t(f) = \sum_{i=1}^{\infty} f(i)\Lambda_t(k)$.

Then M_t satisfies the standard functional equation

$$(3.155)\qquad \begin{aligned} M_t(k,z) = \;& E_k[e^{-c\int_0^t (\widetilde{\zeta}(s))ds}]z \\ & + E_k\left[\int_0^t e^{-c\int_0^s \widetilde{\zeta}(u)du} \cdot c\widetilde{\zeta}(s) \cdot M_{t-s}(\widetilde{\zeta}(s)-1,z) M_{t-s}(1,z) ds\right], \end{aligned}$$

with $M_t(0,z) \equiv 1$. Here $\widetilde{\zeta}$ is a birth and death process with birth rate sk and death rate $\frac{1}{2}dk(k-1)$. The above equation can be solved recursively for the coefficients of z^j.

This relation can also be used to obtain moment formulas by evaluating the appropriate derivatives at $z = 1$.

Consider the semigroup $(T_t)_{t\geq 0}$ of the migration defined by

$$T_t f(k) = E_k[e^{-c\int_0^t \widetilde{\zeta}(s)ds} f(\widetilde{\zeta}(t))], \quad k \in \mathbb{N} \tag{3.156}$$

and let

$$f_\ell(j) := j^\ell, \ \ell = 1, 2, 3, \dots. \tag{3.157}$$

Let for $\ell = 1, 2, 3, \cdots$ and $k \in \mathbb{N}$:

$$m_\ell(t,k) = E_k((\Lambda_t(1_{\{\mathbb{N}\}}))^\ell). \tag{3.158}$$

Then the following moment relations hold:

$$m_1(t,k) = T_t f_0(k) + c\int_0^t T_s\{f_1(\cdot)[m_1(t-s,1) + m_1(t-s,\cdot-1))]\}(k)ds. \tag{3.159}$$

$$\begin{aligned} m_2(t,k) = \quad & T_t f_0(k) + c\int_0^t (T_s\{f_1(\cdot)\cdot[(m_2(t-s,\cdot-1) + m_2(t-s,1)) \\ & +(m_1(t-s,1)\cdot m_1(t-s,\cdot-1))]\}(k)ds. \end{aligned} \tag{3.160}$$

Also note that $m_1(t,k+j) \leq m_1(t,k) + m_1(t,j)$ so that $m_1(t,k) \leq k m_1(t,1)$. Therefore we get from (3.159)

$$m(t,1) \leq T_t 1 + c\int_0^t (T_s k)(1)(m_1(t-s,1)ds + c\int_0^t T_s(k(k-1)m_1(t-s,1))(1)ds. \tag{3.161}$$

Let $M(\lambda)$, $F(\lambda)$, $G(\lambda)$ and denote the Laplace transforms of m_1, $T_s 1$, $T_s k + T_s(k(k-1))$ as functions of s, respectively.

Then

$$M(\lambda)(1 - cG(\lambda)) \leq F(\lambda). \tag{3.162}$$

We need $\lambda > \lambda_0$ so that this is invertible. This gives an upper bound for the Malthusian parameter α.

Step 2 *Stable age and size distribution*

Return to the McKean-Vlasov version of the dual process (which is the dual in the collision-free regime in the limit $N \to \infty$). If we consider for every time t for each individual (i.e. occupied site in our case) currently alive its age and size, then we can introduce a random probability measure, the normalized empirical age and size distribution of the current population, which we denote by

$$\mathcal{U}(t,du,j) = \frac{\Psi(t,du,j)}{K_t}. \tag{3.163}$$

The marginal random measure $\mathcal{U}(t,du,\mathbb{N})$ converges in law (we use the weak topology on measures) as $t \to \infty$ to a *stable age distribution*

$$\mathcal{U}(\infty,du,\mathbb{N}) \text{ on } [0,\infty), \tag{3.164}$$

according to Corollary 6.4 in [N], if condition 6.1 therein holds. The condition 6.1 in [N] or (3.1) in [JN] requires that (with μ as in (3.135)):

$$\int_0^\infty e^{-\beta t}\mu(dt) < \infty \quad \text{for some } \beta \geq 0. \tag{3.165}$$

The condition 7.1 in [N] holds for any $\beta > 0$ since the local (at one site) population of the dual McKean-Vlasov particle system $(\widetilde{\eta}_t)_{t\geq 0}$ given by $(\zeta_{0,t})_{t\geq 0}$ goes to a finite mean equilibrium.

Since the distribution of size at a site depends only on the age of the site, it follows that $\mathcal{U}(t, du, \cdot)$ converges to a *stable age and size* distribution, i.e.

$$(3.166) \quad \mathcal{U}(t,\cdot,\cdot) \Longrightarrow \mathcal{U}(\infty,\cdot,\cdot), \text{ as } t \to \infty \text{ in law.}$$

We can now also calculate α and the asymptotic density of the total number of individuals B as follows unsing a law of large number effect for the collection of independent birth and death processes at the occupied sites. Namely the frequencies of specific internal states stabilize in the stable size distribution, while the actual numbers diverge with the order of K_{t_N} as $N \to \infty$. Given site i let $\tau_i \geq 0$ denote the time at which a migrant (or initial particle) first occupies it. Noting that we can verify Condition 5.1 in [N] we have that

$$(3.167) \quad \lim_{t\to\infty} \frac{1}{K_t} \sum_{i=1}^{K_t} \zeta_{\tau_i}(t-\tau_i) = \int_0^\infty E[\zeta_0(u)]\mathcal{U}(\infty, du) = B \text{ (a constant)}, a.s.,$$

by Corollary 5.5 of [N]. The constant B in (3.167) is in our case given by the average number of particles per occupied site and the growth rate α arises from this quantity neglecting single occupation. Namely define

$$(3.168) \quad \alpha = c\sum_{j=2}^{\infty} j\mathcal{U}(\infty,[0,\infty),j) < \infty, \qquad \gamma = c\,\mathcal{U}(\infty,[0,\infty),1).$$

Then

$$(3.169) \quad B = \frac{\alpha+\gamma}{c}.$$

Furthermore the average birth rate of new sites (by arrival of a migrant at an unoccupied site) at time t (in the process in the McKean-Vlasov dual) is equal to

$$(3.170) \quad \alpha = c\,B - \gamma.$$

3.2.6 Dual process in the collision regime: macroscopic emergence Proposition 1.7(a)

We have obtained in Subsubsection 3.2.4 a lower bound on the emergence time by an upper bound on the number of dual particles. Now we need an upper bound on the emergence time via a lower bound on the number of dual particles.

In order to prove (1.67) return to the dual particle system specified below (3.97). Then after the lower bound for the emergence time in Subsubsection 3.2.4, where we ignored collisions in the dual particle system, we derive here a lower bound on the growth of the number of occupied sites in this system starting with one occupied site incorporating the effect of *collisions*. We proceed in five steps, first we consider lower bounds on the number of particles in the dual process, then refine this in a second step by constructing a multicolour particle system which is an enrichment of the dual particle system, then in a third step we prepare the estimation of the difference between collision-free and dual system, and in the fourth step we turn this into an upper bound for the emergence time. In Step 5 we then show the convergence of the hitting times for reaching $\lfloor \varepsilon N \rfloor$ dual particles.

Step 1: Number of sites occupied by the dual process: preparation

Here we must take into account the effect of collisions and show that (3.139) indeed gives the correct α which describes the growth of the number of factors in the dual process.

If we have collisions, that is, migration to occupied sites, we have two effects. (1) We have *interaction* between sites. (2) If a particle migrates from a site containing only one particle to an occupied site, this results in a *decrease* by one in the number of occupied sites. For these two reasons the process counting the number of occupied sites, denoted by K_t^N, is no longer approximated by a nondecreasing Crump-Mode-Jagers process (which is based on independence of internal states) when it reaches size $O(N)$ and at this moment it is also no longer non-decreasing since now singly occupied sites can dissappear by a jump to an occupied site.

To handle these two problems the key idea is to carry out the analysis of the dynamics separately in three time intervals defined as follows. Set first

$$\tau_{\log N} = \inf\{t : K_t = \lfloor \log N \rfloor\}, \quad \tau^N(\varepsilon) = \inf\{t : K_t^N = \lfloor \varepsilon N \rfloor\}. \tag{3.171}$$

Then define the three time intervals as:

$$\begin{aligned}
&[0, \tau_{\log N}), \\
&[\tau_{\log N}, \tau_{\log N} + \frac{1}{\alpha}(\log N - \log\log N)), \\
&[\tau_{\log N} + \frac{1}{\alpha}(\log N - \log\log N), \tau^N(\varepsilon)].
\end{aligned} \tag{3.172}$$

What happens in the *first interval*? Note (since $\tau_{\log N} << \tau^N(\varepsilon)$ for $N \to \infty$) that asymptotically as $N \to \infty$, in a single migration step the probability of a collision is $O(N^{-1} \log N)$. In particular over a time horizon of length $o(N/\log N)$ the probability to observe a collision ever goes to zero as $N \to \infty$. Hence looking at the formulas we obtained for α and with the properties of exponentially growing populations (occupation measure in current size) we can replace our dual with the mean-field dual without collisions so that with (3.143) we can conclude as $N \to \infty$:

$$\tau_{\log N} = \frac{\log\log N}{\alpha} - \frac{\log W}{\alpha} + o(1) \tag{3.173}$$

and we know furthermore that this process has (asymptotically) reached the stable age distribution at time $\tau_{\log N}$ according to the analysis of Subsubsection 3.2.5, Step 2 therein.

Now consider the *second interval*. From the previous upper bound result still no collisions occur comparable to the dual population in this interval (the collision probability is now at most $O((\log N)^{-1})$ and hence at most only finite many collisions may occur over the whole time interval, essentially at the end and we can assume that asymptotically as $N \to \infty$ we can replace our dual process again by the collision free mean-field dual at the left end point but now in the *whole time span* the stable age distribution is in effect. Hence we can in the second interval continue working with the mean-field dual without collisions and we can even assume the stable age distribution in effect.

Returning to our dual process we conclude (recall (3.172)) that at the end of the second interval we have as $N \to \infty$:

$$K^N_{\tau_{\log N} + \frac{1}{\alpha}(\log N - \log\log N)} \sim W \frac{N}{\log N}. \tag{3.174}$$

Remark 20 *Note that this means that $\tau^N_{s_N} - \tau^N_{\log N}$, with $s_N >> \log\log N$ but $s_N << \log N$, is asymptotically as $N \to \infty$ deterministic.*

We now consider the *third interval*, i.e. the regime in which by (3.174) above

$$K_t^N \geq W \frac{N}{\log N}. \tag{3.175}$$

In this interval two new effects must be considered, namely,

(A) a decrease in the number of occupied sites when a lone particle at an occupied site jumps to another currently occupied site

(B) a particle moves from a site occupied by more than one particle and jumps to another occupied site.

We note that effect (A) also increases the age distribution of occupied sites since in a site the process is stochastically increasing and the young sites can disappear therefore easier (and hence the age of the total population becomes stochastically larger). Effect (B) tends to increase the size of occupied sites of a given age compared to the case without collision. Therefore the result of (A) and (B) is to tend to increase the average size of occupied sites compared to a system without collision (recall the older sites are stochastically larger), but decreases the number of occupied sites.

We note that larger sites increase the number of migration steps. Altogether this means that we expect that the rate of growth of K_t^N denoted $\beta_N(t)$ (i.e. $\beta_N(t) = (K^N_{t+\triangle t} - K_t^N)/K_t^N$) satisfies in the third time interval (for $N \geq \log\log N$) if the present state of K_t^N is k:

$$\beta_N(\cdot) \geq \alpha(1 - \frac{k}{N}). \tag{3.176}$$

In order to control these effects precisely we use the technique of coupling constructed from an enriched, i.e. multicolour particle system, which we introduce next.

Step 2: A multicolour particle system

To examine rigorously the growth in the time span, in which the number of occupied sites reaches $O(N)$ we construct a coupled system of *multicoloured particles* such that we can compare the new system with collisions effectively with the simpler one without collisions, and are able to carry out estimates on the difference. This way we can obtain the lower bound on the number of occupied sites and the total number of particles.

First an informal description. We shall have *white, black* and *red* particles. We start with white particles only. In intervals 1 and 2 the process of the white particles grows as before, i.e. without collisions. However as soon as collisions occur we consider a *modified system.* The colouring of the system allows us to keep track of events like (A) and (B) given above. For example in the case of (A) we will mark the lost site by placing there a black particle and the particle that jumped and collided with other particles on the new location is marked by giving it a red colour. We handle the second effect (B) by using bounds from below on the rate of founding of new occupied sites.

Formally proceed as follows. The multicolour comparison system has black, white and red particles. It has white and red particles located at the sites $\{1, 2, \cdots, N\}$ and black particles located at a site in $\mathbb{N}$ where $\{1, 2, \cdots, N\}$ and $\mathbb{N}$ are disjoint finite and countable sets respectively. In other words the geographic space of this new system is

$$\{1, 2, \cdots, N\} + (\mathbb{N}), \tag{3.177}$$

and the state space is

$$(\{1, 2, \cdots, N\} + (\mathbb{N}))^{\mathbb{N}_0^3}. \tag{3.178}$$

The initial state is given by having only white particles, which are located at sites in $\{1, 2, \cdots, N\}$ such that occupation numbers are exchangeable on this part of the geographic space.

The dynamics of the new system is markovian more precisely it is a pure Markov jump process and we specify the transitions and their rates. Instead of writing the generator we describe this more intuitively in words. This runs as follows:

- *white* particles at a site follow the same local dynamics as the dual particle system η as far as birth (of white particles) and coalescence (of white particles) goes, changes occur for migration.

  Let k denote the number of sites having currently at least one white particle. Each migrating white particle moves with probability $1 - \frac{k}{N}$ to a new site in $\{1, 2, \cdots, N\}$ which prior to this event did not contain any white particles and with probability $\frac{k}{N}$ changes to a black particle now located at a new unoccupied site in $\mathbb{N}$ and at the same time also a red particle is produced at an occupied site in $\{1, 2, \cdots, N\}$ chosen at random among the k occupied sites.

  Hence a migrating white particle produces:

  (3.179) a new site occupied with a white particle with probability $(1 - \dfrac{k}{N})$,

  and two particles,

  (3.180) one black and one red, with probability $\dfrac{k}{N}$,

  (3.181) the red particle is placed at a randomly chosen occupied site in $\{1, 2, \cdots, N\}$, the black one at the smallest free site in $\mathbb{N}$.

- *Red* particles have the same dynamics as the dual particle system η on $\{1, 2, \cdots, N\}$ (newborn particles are also red) and in addition when a red and white at the same site coalesce the outcome is always white.

- *Black* particles follow the same dynamics as the white except that migrating black particles move on $\mathbb{N}$ and always go to a new, so far unoccupied site in $\mathbb{N}$.

This means that no collisions (by migration!) occur among white particles and furthermore by the collision convention the white particles are not influenced by the presence of red particles, nor are they influenced by the black particles. Hence the key observation about the new system is that:

- The number of occupied sites in the *union* of the *black and white* particles follows the dynamics of the process without collisions, i.e. the number of sites they occupy is a version of $(K_t)_{t \geq 0}$,

- the number of occupied sites in the *union* of the *white and red* particles follows the exact dual dynamics, i.e. the number of sites they occupy is producing a version of $(K_t^N)_{t \geq 0}$.

- The process of white and black particles per site follow the dynamics of ζ and the one of white and red particles per site that of ζ^N.

- We have a coupling of ζ and ζ^N given by the embedding in the multicolour system and the difference process $\zeta_t - \zeta_t^N$ can be represented as the number of black minus the number of red particles at the various sites. In particular also K, K^N and $K - K^N$ can be represented in terms of the multicolour system.

This construction allows to give upper and lower bounds on the time to reach with the dual process $\lfloor \varepsilon N \rfloor$ occupied sites, which then in turn gives bounds on the number of factors in the dual process.

Since we know from the last subsubsection how the growth of the population of both black and white particles works, we use this as an upper bound on the number of white particles (conditioned on W).

We denote the *number of sites occupied only by black particles* (recall "only" is by the construction not a constraint), respectively the number of sites occupied by either *white or black particles* at time t by

(3.182) $\quad \bar{Z}_t^N, \text{ respectively } \widehat{K}_t.$

Note that $\widehat{K}$ is a version of K.

We shall get below an upper bound on the number of sites occupied by black particles and then we can use this upper bound to get a *lower bound* on the number of sites occupied by white particles and therefore a lower bound for the number of factors of the dual (red plus white particles).

Let

(3.183) $\quad \widetilde{\tau}^N(\varepsilon) = \inf\{t : \widehat{K}_t \geq \varepsilon\, N\}.$

This means in particular that

(3.184) $\quad \widetilde{\tau}^N(\cdot)$ is nondecreasing.

(Note the difference between $\widetilde{\tau}^N(\varepsilon)$ and $\tau^N(\varepsilon)$ is that we use in the first case white and *black* particles and in the second case white and *red* particles. In addition $\widehat{K}$ is a version of K.)

We just saw above that we have to estimate the number of black particles to compare the dual and the dual of the McKean-Vlasov limit which differ by the difference of black and red particles. The latter is estimated above stochastically by the number of black particles. This holds since if we associate the black and red upon their creation, we see that the red ones have an extra chance to disappear once they coalesce with a white particles or collide with red particles from another black-red creation event.

Step 3: Estimating the number of black particles.

An upper bound on the number of black sites (recall black particles sit on $\mathbb{N}$, while white and red sit on $\{1, \cdots, N\}$) at time $\widetilde{\tau}^N(\varepsilon)$, is obtained as follows. Bound the number of migration steps in the model by a Crump-Mode-Jaegers process which is in distribution given by the process of black and white particles. This means we construct up to terms of order $o(N)$ a stochastic upper bound for the number of black particles by realizing *independently* a Poisson stream of potential collision events according to the law of a driving CMJ-process generating potential new sites and with a collision probability given by the current number of white sites divided by N.

Therefore an upper bound on the production rates of new black populations is obtained by integrating the production rate of new sites generated by the process of white and black particles at a given time (which is for large times converging to α by (3.136)). Furthermore note that in state k (meaning the number of sites with at least one white particle) with a migration step of a white particle, the probability of a collision and hence the probability to become black is k/N. This gives at time s with the CMJ-theory a production rate of collisions which is stochastically bounded by

(3.185) $\quad \alpha \dfrac{W_s e^{\alpha s}}{N} W_s e^{\alpha s}.$

Hence we get a stochastic bound on the intensity of migration steps resulting in collisions of the form $W_s^2 \alpha (e^{\alpha s}/N) e^{\alpha s}$. This expression therefore bounds the rate of creation of *founders* of new black families of particles.

We can replace for $s \geq \tau_{\log N}$, the quantity W_s^2 in the limit $N \to \infty$ by W^2, due to the convergence theorem for CMJ-processes. Therefore we can estimate the rate at which new black families are created asymptotically as $N \to \infty$ and work with the expression

(3.186) $\quad W^2 \alpha e^{2\alpha s}/N \text{ for } s \geq \tau_{\log N}.$

The contribution to the stream of production of collisions arising before time $\tau_{\log N}$ goes to zero in probability as $N \to \infty$ and therefore the probability that black families are founded before time $\tau_{\log N}$ tends to zero as $N \to \infty$, more precisely at rate $(\log N)/N$.

In principle the time we consider, i.e. from $\tau_{\log N}$ to $\widetilde{\tau}_N(\varepsilon)$ seems random on first sight, however it is asymptotically deterministic. Namely $\widetilde{\tau}_N(\varepsilon) - \tau_{\log N}$ becomes deterministic in the sense that

$$(3.187) \quad (\widetilde{\tau}_N(\varepsilon) - \tau_{\log N}) - (\frac{1}{\alpha}\log(\frac{\varepsilon N}{W}) - \frac{1}{\alpha}\log(\frac{\log N}{W})) \underset{N\to\infty}{\longrightarrow} 0, \text{ a.s.}$$

so that conditioned on W as $N \to \infty, \widetilde{\tau}_N(\varepsilon) - \tau_{\log N}$ can be replaced by a deterministic quantity. This allows us to generate as upper bound on the number of black particles the following process.

(1) Realise a Poisson point process with intensity measure

$$(3.188) \quad (W^2\alpha e^{2\alpha s}/N)ds.$$

(2) Then let from each point of this Poisson point process evolve independent families of black particles with the usual dynamics but independent of the Poisson point process.

Next observe that a new black founding particle generated by the Poisson point process has a *descendant black population* which forms by construction a Crump-Mode-Jagers process with growth rate α, i.e. it grows like $\widetilde{W}_{(\widetilde{\tau}^N(\varepsilon)-s)}e^{\alpha(\widetilde{\tau}^N(\varepsilon)-s)}$, if s is the birth time of the black particle and provided we observe up to the final time $\widetilde{\tau}^N(\varepsilon)$. Here $\{\widetilde{W}_s\}_{s\in\mathbb{R}}$ are independent copies of W.

The newly founded black populations created at the Poisson point process jump times observed at time $\widetilde{\tau}_N(\varepsilon)$ have sizes

$$(3.189) \quad \{\widetilde{W}_{(\widetilde{\tau}_N(\varepsilon)-s),N}\exp(\alpha(\widetilde{\tau}_N(\varepsilon)-s)), \quad s \in \{s_1^N, s_2^N, \cdots, s_{M(N)}^N\}\},$$

(we will suppress the subscript N for s_i^N in formulas below) and are independently distributed conditioned on the process of birth-times and $\widetilde{\tau}_N(\varepsilon)$. For these copies we have a law of large numbers acting. The claim is more precisely that if we take the expectation $\widetilde{E}$ over $\{\widetilde{W}_{\widetilde{\tau}_N(\varepsilon)-s_i,N}, \quad i \in \mathbb{N}\}$ then as $N \to \infty$ we have the following law of large numbers effect:

$$(3.190) \quad \sum_{i=1}^{M(N)} \widetilde{W}_{\widetilde{\tau}_N(\varepsilon)-s_i,N}\exp(\alpha(\widetilde{\tau}_N(\varepsilon)-s_{i,N})) \sim \int_0^{\widetilde{\tau}_N(\varepsilon)} \alpha W^2(e^{2\alpha s}/N)\widetilde{E}[\widetilde{W}_{(\widetilde{\tau}_N(\varepsilon)-s),N}]e^{\alpha(\widetilde{\tau}_N(\varepsilon)-s)}ds$$

and the r.h.s. is asymptotically as $N \to \infty$ equal to

$$(3.191) \quad E[W]\int_0^{\widetilde{\tau}_N(\varepsilon)} \alpha W^2(e^{2\alpha s}/N)e^{\alpha(\widetilde{\tau}_N(\varepsilon)-s)}ds,$$

using the convergence theorem for the Crump-Mode Jagers process.

Then combining (3.189) and (3.190) together with (3.191) we obtain a *mean conditioned on W* of the l.h.s. of (3.190) which is asymptotically as $N \to \infty$ equal to:

$$(3.192) \quad [\int_{\tau_{\log N}}^{\widetilde{\tau}^N(\varepsilon)} E[W]\alpha W^2 e^{\alpha\widetilde{\tau}_N(\varepsilon)} \cdot e^{\alpha s}ds] \sim E[W]W^2\frac{1}{N}(\frac{1}{W^2}\varepsilon^2N^2) = E[W]\varepsilon^2N.$$

To justify the law of large numbers given in (3.190), we observe that the number of birth events of black populations goes to zero for times t_N with

$$(3.193) \quad t_N - \frac{\log N}{2\alpha} \underset{N\to\infty}{\Longrightarrow} -\infty$$

and for these times the order of magnitude of a descending population is at most of order $\sqrt{N} = o(N)$. Therefore we get a diverging number of contributions which are all $o(N)$.

More precisely, the law of large numbers is verified by showing that

$$(3.194) \quad \widetilde{V}ar\left[\sum \widetilde{W}_{(\widetilde{\tau}_N(\varepsilon)-s_i),N} e^{\alpha(\widetilde{\tau}_N(\varepsilon)-s_i)}\right] = O(\frac{N^2 \log N}{N^3}) = O(\frac{\log N}{N}).$$

Here we use the explicit representation and the fact that $Var(W_s) < \infty$ and $Var(W_s) \to Var(W)$ as $s \to \infty$ together with the asymptotics of $\widetilde{\tau}_N(\varepsilon)$. This concludes the proof for the law of large numbers.

Hence we can bound the number of black particles by the asymptotic growth in N of its expectation over the randomness in the process of growth, i.e. the $\widetilde{W}_{s_i^N}$ in the black population.

This reasoning gives the following asymptotic stochastic upper bound on the number of black sites, namely the expression:

$$(3.195) \quad W^2 \int_{\tau_{\log N}}^{\widetilde{\tau}^N(\varepsilon)} \alpha \frac{e^{\alpha s}}{N} e^{\alpha s} E[\widetilde{W}_s] e^{\alpha(\widetilde{\tau}^N(\varepsilon)-s)} ds \lesssim E[W]N\varepsilon^2, \text{ as } N \to \infty,$$

using the upper bound on $\widetilde{\tau}^N(\varepsilon)$ implied by (3.143).

Therefore we conclude that the *proportion of black sites* among all sites in the black and white system at time $\widetilde{\tau}^N(\varepsilon)$ is at most (in the limit $N \to \infty$) equal to

$$(3.196) \quad \frac{E[W]}{W}\varepsilon.$$

In particular by letting $\varepsilon \to 0$ this relative frequency goes to 0.

In particular this would imply that the calculation below of $\tau^N(\varepsilon)$ (the time for the number of white plus red sites to reach ε) will give a stochastic lower bound for

$$(3.197) \quad \tau^N(\varepsilon - const \cdot \varepsilon^2),$$

if we condition on W.

Step 4: Upper bound on emergence time

Recalling (3.176) we see that given the current state k of K_t^N, the time of the next birth in K_t^N is a positive random variable denoted τ_k for which we are waiting for the next migration step of a white or red particle to an unoccupied site. This waiting time is *bounded above* by the waiting time of a white particle migration step to an unoccupied site.

Observe that due to the uniform migration distribution, the property to jump to an unoccupied site is an experiment independent of everything else and the jump is successful with probability $1 - k/N$ if k is the current number of occupied sites. Therefore we can view the formation of new occupied colonies as a *pruning* of the point process of white particle migration steps.

What can we say about the point process of the attempted migration steps by white particles? We note first that this migration step is *given all current internal states* the infimum of k, independent exponential waiting times, each of which is the minimum of exponential waiting times for the internal transitions like birth, death and emigration. Once an internal transition other than migration occurs, the experiment starts over with this new internal states. The rates of all this exponential waiting times depend on the internal state in the corresponding site. It is best to view this as trials consisting of migration steps arising from k waiting times built from exponentials according to the internal states, waiting for one which leads to a new colony.

We know from the CMJ-theory that the number of migration steps in the white plus black system, with k occupied at time t, in the time interval $[t, t + \triangle t]$ behaves asymptotically as $\alpha k \triangle t$

as $k \to \infty$ a.s. and in L_1 as long as k has not reached too large values. We now use the observation that the relative frequency of black colonies is asymptotically as $N \to \infty$ negligible so that asymptotically the white migration steps are as frequent as the ones of the white and black system as long as k is large but k/N remains small.

When the current number of occupied sites is k the waiting time for the next colonization has mean *bounded above* by that of the waiting time for the birth of white site. To obtain information on this random time we note that the next migration of a particle is given by an exponential random clock which occurs with a hazard function given by the rate $c\sum_{i=1}^k \zeta_i(\cdot)$ and the probability that a migration produces a new white site is $1-\frac{k}{N}$ (compare (3.176)). To get an upperbound on the time $\tau_{\log N}$ we note that $\zeta_i(\cdot) \geq 1$ and note that the number of migration events needed to obtain a new white site is geometric with mean $(1-k/N)^{-1}$.

Recall that $\tau_{\log N}$ is given by (3.173). Then recalling the convergence to the stable age and size distribution for $t \to \infty$ for $k \to \infty$ and therefore $\frac{c}{k}\sum_{i=1}^k \zeta_i(\cdot) \sim \alpha$, we argue below in detail that the expected time, given the internal states of the occupied sites to obtain the next white site for $\log N \leq k \leq \varepsilon' N$, behaves for $N \to \infty$ as follows:

$$(3.198)\qquad \frac{1}{k(\frac{1}{k}\sum_{i=1}^k 1_{\zeta_i>1}\zeta_i(\cdot))(1-\frac{k}{N})} \sim \frac{1}{k\alpha(1-\frac{k}{N})} = \frac{1}{\alpha k} + \frac{1}{\alpha N}\frac{1}{1-\frac{k}{N}} \leq \frac{1}{\alpha k} + \frac{1}{\alpha N(1-\varepsilon')}.$$

To make more precise the first step, note first that

$$(3.199)\qquad \lim_{N\to\infty}\Big(\frac{1}{K^N_{\tau_{\log N+j}}}\sum_{i=1}^{K^N_{\tau_{\log N+j}}} 1_{\zeta_i>1}\zeta_i(\cdot)\Big) = \lim_{N\to\infty}\Big(\sum_{k=2}^{\infty} kU^N(\tau_{\log N+j},k)\Big).$$

We know from the CMJ-theory that:

$$(3.200)\qquad |\lim_{N\to\infty}\sum_{k=2}^{\infty} kU^N(\tau_{\log N+j},k) - \sum_{k=2}^{\infty} kU(\infty,k)| = 0.$$

Therefore

$$(3.201)\qquad |\lim_{N\to\infty}\sum_{k=2}^{\infty} kU^N(\tau_{\log N+j},k) - \alpha| = 0 \quad \text{uniformly in } j.$$

Secondly we now have to deal with the fact that the quantity we handled above appears in the denominator and is random, so that taking *expectation* needs some care. Note that $\sum_{k=2}^{\infty} kU^N(\tau_{\log N}+j),k) \geq 1-U^N(\tau_{\log N+j},1)$ and

$$(3.202)\qquad \lim_{N\to\infty}(1-U^N(\tau_{\log N+j},1)) > \frac{1}{2}(1-U(\infty,1)) > 0.$$

We treat below the event $(1-U^N(\tau_{\log N+j},1)) \leq a$ for small a as a rare event and obtain a large deviation bound for it allowing us to proceed for $N \to \infty$ with assuming we are in the complement.

We break the details of the argument down in two parts, namely, the probability that more than $\frac{\log N}{2}$ of the individuals have age less than or equal to B, that is, $\frac{1}{2}\log N$ particles produce $\frac{1}{2}\log N$ new particles in time $\leq B$. We choose B so that the expected number of particles produced by one site in time B is $\leq \frac{1}{3}$. The second part is to calculate that more than a fraction $(1-a)$ of the sites of age $> B$ have only a single particle where $a > 0$ is chosen so that if a site is of age $\geq B$ then $P(\zeta \geq 2) > 2a$. Since the events that $\frac{\log N}{2}$ particles produce $\frac{\log N}{2}$ particles in time B and the event that more than a fraction $(1-a)$ sites of age B have $\zeta = 1$ are both "rare" events, we can verify the exponential bound:

$$(3.203)\qquad P[(1-U^N(\tau_{\log N+j},1)) \leq a] \leq e^{-C(\log N+j)} \text{ for large } N, \text{ for some } C > 0.$$

We now take expectations *conditioned* to be on the good event,

$$(3.204)\quad E_*[\cdot] = E[\cdot|(1-U^N(\tau_{\log N+j},1)) > a \text{ for } j=1,\dots,\varepsilon N-\log N].$$

We can then conclude (by bounded convergence) that:

$$(3.205)\quad \lim_{N\to\infty} E_*[\frac{1}{\sum_{k=2}^\infty kU^N(\tau_{\log N+j},k)}] \le \frac{1}{\alpha}, \quad \text{uniformly in } j.$$

This means that we now can conclude with T_k denoting the waiting time for the next migration step if we have k occupied sites satisfies:

$$(3.206)\quad E_*[T_k] \le \frac{1}{\alpha k} + \frac{1}{\alpha N(1-\varepsilon')}, \quad k\in[\log N, \varepsilon' N].$$

Remark 21 *For the case considered in Section 8 it is convenient to note the following simple generalization. Consider a system of k independent exponential clocks $C_1,\dots,C_k$ with exponential waiting times with parameters $c_1,\dots,c_k$. Then*

$$(3.207)\quad \begin{aligned} P(\min_i C_i \le \frac{x}{k}) &= (1-\prod_i P(C_i > \frac{x}{k})) \\ &= (1-\prod_i e^{-c_i x/k}) = 1-e^{-(\frac{1}{k}\sum_{i=1}^k c_i)x}) \qquad \text{as } k\to\infty. \end{aligned}$$

In our case $c_i = \zeta_i \in \mathbb{N}$ and this leads to $1-e^{-(\sum_k k\cdot U^N(\tau_{\log N+j},k))}$, and therefore mean $\frac{1}{k\cdot\sum_k k\cdot U^N(\tau_{\log N+j},k)}$ and second moment $2\left(\frac{1}{k\cdot\sum_k k\cdot U^N(\tau_{\log N+j},k)}\right)^2$ and so the variance is $\left(\frac{1}{k\cdot\sum_k k\cdot U^N(\tau_{\log N+j},k)}\right)^2$.

In order to conclude the argument we next choose $\varepsilon > 0$. Then for sufficiently large N, the mean time, $E_*[\tau^N(\varepsilon)]$, for K_t^N required to reach εN (cf. (3.206)) satisfies

$$(3.208)\quad \begin{aligned} E_*[\tau^N(\varepsilon)] \le \qquad & E[\tau_{\log N}] + \frac{1}{\alpha}[\frac{1}{\log N} + \dots + \frac{\log N}{N}] \\ & +\frac{1}{\alpha}[\frac{\log N}{N} + \dots + \frac{1}{\varepsilon N}] + \frac{\varepsilon}{\alpha(1-\varepsilon)}. \end{aligned}$$

Hence

$$(3.209)\quad E_*[\tau^N(\varepsilon)|W] \le \frac{\log N}{\alpha} + c(\varepsilon),$$

where $c(\varepsilon)$ is a constant depending on ε and W.

We also have from the CMJ-theory the lower bound

$$(3.210)\quad E[\tau^N(\varepsilon)|W] \ge E[\widetilde{\tau}^N(\varepsilon)|W] = \alpha^{-1}\log N - \left(\frac{|\log\varepsilon'| + \log W}{\alpha}\right).$$

Therefore it follows that $\tau^N(\varepsilon)$, conditioned on W, is of the form $\alpha^{-1}\log N + O(1)$, as claimed. We also see that the difference between $\tau^N(\varepsilon)$ and $\widetilde{\tau}^N(\varepsilon)$ is a random variable satisfying

$$(3.211)\quad E_*[|\tau^N(\varepsilon) - \widetilde{\tau}^N(\varepsilon)|] < \infty.$$

We can also estimate now the time till fixation as follows. The coupling construction now implies that if we choose ε small enough, then at time $\alpha^{-1}\log N$ we will reach εN dual particles

which will then in finite random time produce a mutation jump so that we get indeed (1.67) and thus completing the argument. Namely since

$$\Pi_t^N \geq \widehat{K}_t - \bar{Z}_t^N, \quad \bar{Z}_t^N \leq \text{ r.h.s. (3.195)}, \tag{3.212}$$

after time $\widetilde{\tau}^N(\varepsilon)$ there is a mutation rate at least

$$m(\varepsilon - E[W]\varepsilon^2)^+ \tag{3.213}$$

uniformly in large N and therefore the probability (recall $\tau^N(\varepsilon) - \widetilde{\tau}^N(\varepsilon) = O(1)$ as $N \to \infty$) that a mutation does not occur before $\tau^N(\varepsilon) + t$ goes to zero as $t \to \infty$ uniformly in N if we choose ε small enough. This completes the proof of (1.67).

Step 5: Extension: convergence of hitting times

However we can get even more mileage from the above argument to make it clear that the randomness in the evolution of spatially macroscopic variables is created only in the very early stages of the growth of the dual particle system.

We next estimate the variance of $\tau^N(\varepsilon) - \tau^N_{\log N}$, a random variable of which we know from the analysis of Step 4 above that it is asymptotically close to $\widetilde{\tau}^N(\varepsilon) - \widetilde{\tau}^N_{\log N}$ as $N \to \infty$ and $\varepsilon \to 0$. If we now condition on all the internal states of all occupied sites we can calculate the variance of the waiting time T_k for the next migration step explicitly since we know it is exponential. Hence we can, denoting by $\widetilde{V}ar(\cdot)$ this conditional variance, estimate $\widetilde{V}ar(\tau^N(\varepsilon) - \tau^N_{\log N})$. Conditioned on W and restricted to the complement of the rare event introduced in Step 4 (which is asymptotically negligible (3.203)) we can verify by explicit calculation (following the same argument as for the means in (3.208) now for the variances, in this case we get the sum of squares of the terms we had in the previous mean calculation) that with $\widetilde{V}ar_*$ denoting this conditional variance *restricted to the complement of the rare event*, we get the bound

$$\limsup_{N\to\infty} \widetilde{V}ar_*[(\tau^N(\varepsilon) - \tau_{\log N})] \leq \limsup_{N\to\infty} \left(\text{const} \cdot \left(\frac{1}{\log \mathrm{N}} - \frac{1}{\varepsilon \mathrm{N}}\right)\right) = 0. \tag{3.214}$$

Now we note that the term in the limsup on the r.h.s. is independent of the internal states and we conclude that conditional on W the (restricted) variance of the difference between $\tau^N(\varepsilon)$ and $\tau_{\log N}$ goes to zero and hence that is, the variance of the sum conditioned on W of the waiting times (with means given in (3.198)) in the second and third time intervals given in (3.172) goes to 0 as $N \to \infty$ since the rare event (and hence the restriction) become negligible as $N \to \infty$. The analogous result also holds for $\widetilde{\tau}^N(\varepsilon) - \widetilde{\tau}_{\log N}$.

This means that the growth in the second and third time interval we have specified in (3.172) is *deterministic* and the randomness occurred earlier in the first time interval.

Note that during the first time interval as $N \to \infty$ the expected number of collisions is $O(\frac{(\log N)^2}{N})$ so that no collisions occur with probability tending to 1. Therefore as $N \to \infty$,

$$\widetilde{\tau}^N_{\log N} - \tau^N_{\log N} \underset{N\to\infty}{\longrightarrow} 0 \quad \text{in probability.} \tag{3.215}$$

Together with the fact that as $N \to \infty$ the limiting probability of the rare event is 0 it follows that there exists a deterministic sequences, namely

$$(\widetilde{\delta}^\varepsilon_N = E[\widetilde{\tau}^N_{\varepsilon N} - \widetilde{\tau}^N_{\log N}])_{N\in\mathbb{N}}, \, (\delta^\varepsilon_N = E_*[\tau^N_{\varepsilon N} - \tau^N_{\log N}])_{N\in\mathbb{N}} \tag{3.216}$$

such that for $\eta > 0$,

$$P[|\widetilde{\tau}^N(\varepsilon) - \widetilde{\tau}^N_{\log N} - \widetilde{\delta}^\varepsilon_N| > \eta] \to 0, \tag{3.217}$$

and

$$P[|\tau^N(\varepsilon) - \tau^N_{\log N} - \delta^\varepsilon_N| > \eta] \to 0. \tag{3.218}$$

We now collect the above estimates to prove the following.

Proposition 3.5 *(Convergence of hitting times of level ε)*
There exists a non-degenerate $\mathbb{R}$-valued random variable $\widetilde{\tau}(\varepsilon)$ such that

$$\widetilde{\tau}^N(\varepsilon) - \frac{\log N}{\alpha} \Rightarrow \widetilde{\tau}(\varepsilon), \text{ as } N \to \infty, \tag{3.219}$$

$$\tau^N(\varepsilon) - E_*[\tau^N(\varepsilon)] \text{ converges in distribution} \tag{3.220}$$

and there exists a constants $\{C(\varepsilon)\}_{\varepsilon>0}$ such that $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$ and

$$\lim_{N\to\infty} P\Big(\frac{\log N + \log\varepsilon - \log W}{\alpha} - C(\varepsilon) < \tau^N(\varepsilon) < \frac{\log N + \log\varepsilon - \log W}{\alpha} + 2C(\varepsilon)\Big| W\Big) = 1. \quad \square \tag{3.221}$$

Proof We begin with some preparatory points. Note that asymptotically as $N \to \infty$ by (3.143),

$$\widetilde{\tau}^N_{\log N} - \frac{\log\log N - \log W_N}{\alpha} = o(1) \tag{3.222}$$

where $W_N \to W$ a.s., and by (3.215) it follows that

$$\tau^N_{\log N} - \frac{\log\log N - \log W_N}{\alpha} = o(1). \tag{3.223}$$

But by (3.217), (3.218),

$$P[|(\tilde{\tau}^N(\varepsilon) - \widetilde{\tau}_{\log N} - \widetilde{\delta}^\varepsilon_N) - (\tau^N(\varepsilon) - \tau^N_{\log N} - \delta^\varepsilon_N)| > 2\eta] \to 0. \tag{3.224}$$

But then (3.211) implies that

$$\sup_N |\widetilde{\delta}^\varepsilon_N - \delta^\varepsilon_N| < \infty. \tag{3.225}$$

We also have that $\widetilde{\delta}^\varepsilon_N - (\frac{\log(N\varepsilon) - \log\log N}{\alpha}) \to 0$. Therefore as $N \to \infty$:

$$\tilde{\tau}^N(\varepsilon) - \frac{\log N}{\alpha} \to \frac{\log\varepsilon - \log W}{\alpha}. \tag{3.226}$$

Having finished the preparation we already see that (3.226) implies the weak convergence claimed in (3.219). Now we come to the proof of (3.220) and (3.221).

The relation (3.221) then follows from using (3.234) and inserting (3.223), (3.222) together with (3.225) and the fact that $\sup_N |\delta^\varepsilon_N - \widetilde{\delta}^\varepsilon_N|$ tends to zero as $\varepsilon \to 0$. In order to see now (3.220) we have to use that given W the $\tau^N(\varepsilon) - \tau^N_{\log N}$ is as $N \to \infty$ asymptotically deterministic as we saw above. Then the claim follows with (3.223).

3.2.7 Dual process in the collision regime: nonlinear dynamics

In order to obtain more detailed results on the emergence regime and on the process of fixation via the dual process we will need more precise information about the growth of the number of sites occupied by the dual process, respectively the states at these sites, once we reach the regime where *collisions* play a role.

The behaviour in this regime is qualitatively very different from the collision-free situation and is the result of three properties of the evolution mechanism:

1. Randomness enters in early stages of the growth of the dual process, i.e. times in $[0, O(1)]$, here the dual behaves as a CMJ branching process described by *linear* (i.e. Markov) but *random* dynamics,

2. in the time regime

$$(3.227) \qquad [s(N), \frac{\log N}{s(N)}), \text{ with } s(N) \to \infty,\ s(N) = o(\log N)$$

the law of large numbers is in effect leading to *deterministic* but still *linear* dynamics

3. in the regime ($s(N)$ as above)

$$(3.228) \qquad [\frac{\log N}{s(N)}, \frac{\log N}{\alpha} + t)$$

the dynamics is *deterministic* but the evolution equation becomes *nonlinear* due to collisions.

The problem is now that these regimes separate as $N \to \infty$ and we have to connect them through a careful analysis. To integrate all three phases into a *deterministic nonlinear* dynamic with *random* initial condition, the full dual process for the finite system with N exchangeable sites is analysed.

Indeed in this and the next subsubsections we show that in the limit $N \to \infty$ the initial asymptotically collision-free evolution stage in (1) and (2) produces *a random initial condition* for the nonlinear dynamics arising in (3) where the randomness arises from stage (1). A key step in linking the random and nonlinear aspects is the analysis of the evolution in the time interval in (3.228) in the limit $N \to \infty,\ t \to -\infty$.

In this subsection we consider the time interval

$$(3.229) \qquad \frac{1}{\alpha}[\log N - \log\log N, \log N + T)$$

in which the dynamics are deterministic. Recall that

$$(3.230) \qquad u^N(\frac{1}{\alpha}(\log N - \log\log N)) = W_N \frac{N}{\log N} = W \frac{N}{\log N} + o(1) \text{ as } N \to \infty,$$

since $W_N \Rightarrow W$ by the equation (3.143).

We have to describe the dual process in the above time interval in the limit $N \to \infty$ and to obtain a limit dynamic which allows to draw the needed conclusions for the original process. We proceed in three steps. It turns out that the main properties of the dual process needed for that purpose can be captured in a triple of functionals of this process which we introduce in Step 1. In Step 2 we formulate the corresponding limiting (for $N \to \infty$) objects which allows us to finally state and prove in Subsubsection 3.2.8 the key convergence relation for the dual process. Step 3 proves that the defined evolutions have the desired properties.

Step 1 *Some functionals of the dual in the time span up to fixation*

First, we observe that to determine the quantity of interest for emergence namely

$$(3.231) \qquad \frac{\Pi^N_{T_N+t}}{N} \quad , \text{ for } T_N = \frac{\log N}{\alpha}$$

as a function of t we need to know the number of sites occupied at time $T_N + t$ and relative frequencies of the occupation numbers of these sites. Since the occupation number of a site depends on the age of the site, we will keep track of the number of occupied sites and the relative frequencies

of both age and occupation number for these sites. As $N \to \infty$ a law of large number argument is then used to obtain the desired information.

We consider the number of sites occupied at time t denoted K_t^N and the corresponding measure-valued process on $[0,\infty) \times \mathbb{N}_0$ giving the unnormalized number of sites of a certain age u in $[a,b)$ and occupation size j:

$$\Psi^N(t,[a,b),j)) = \int_{(t-b)}^{(t-a)} 1_{(K_u^N > K_{u-}^N)} 1_{(\zeta_u^N(t)=j)} dK_u^N, \tag{3.232}$$

where $\zeta_u^N(t)$ denotes the occupation number at time t of the site born at time u, that is, a site first occupied the last time at time u, which is therefore at time t exactly of age $t-u$.

The *normalized empirical age and size distribution* among the occupied sites is defined as:

$$U^N(t,[a,b),j) = \frac{1}{K_t^N} \Psi^N(t,[a,b),j), \quad t \geq 0,\ j \in \{1,2,3,\dots\}. \tag{3.233}$$

Thus for $(\zeta_t^N)_{t \geq 0}$ we have a growing number K_t^N of different interacting birth and death processes which each have in state k linear birth rate sk, linear death rate $c(1-\frac{1}{N})k$, quadratic death rate $dk(k-1)$. In addition immigration of one additional element into a site of size k occurs with rate

$$c\left(\frac{K_t^N c^{-1}(\alpha_N(t)+\gamma_N(t))-k}{N}\right), \tag{3.234}$$

where $\alpha_N(t)$ is c times the mean size of sites with more than one particle, γ_N is c times the frequency of singletons; these will be defined precisely below in (3.239). Note that $U^N(t,dx,j)$ is a pure atomic measure which evolves with jumps of size $\frac{1}{K_t^N}$ one for each migration event to a yet unoccupied site, or $-\frac{1}{K_t^N}$ when a singly occupied site migrates and a collision occurs.

Remark 22 *Note that the $\zeta_u^N(\cdot)$ interact if we are in a regime where collisions occur, that is at times after emergence has occurred which happens at times of order $\frac{\log N}{\alpha}$. The marginal distribution of age is the analogue of the empirical age distribution in (3.163). However now we take into account possible collisions, that is, a migration to a previously occupied site when we consider the system of N sites with symmetric migration among these sites and in this case the distribution of $\zeta_u^N(t)$ may depend not only on the age $t-u$ but also on t due to changes in the collision rates.*

Set for convenience to describe the pair given by the number of sites and size-age distribution:

$$u^N(t) := K_t^N. \tag{3.235}$$

We have obtained with (3.233), (3.235) now a pair which is $\mathbb{N} \times \mathcal{P}([0,\infty) \times \mathbb{N})$-valued, denoted

$$(u^N(t), U^N(t,\cdot,\cdot))_{t \geq 0} \tag{3.236}$$

and which completely describes our dual particle system up to permutations of sites. We can recover the functional Π_t^N as

$$\Pi_t^N = u^N(t) \sum_{j=1}^{\infty} j U^N(t,[0,t],j). \tag{3.237}$$

However since we work with exchangeable initial states for our original population process, the pair (u^N, U^N) provides a sufficiently complete description of the dual process to allow us to calculate the laws of our original processes.

In the interval $\alpha^{-1}[\log N - \log\log N, \log N + T]$ the process $(u^N(t))_{t\geq 0}$ increases by one, respectively decreases by one, at rates

$$\alpha_N(t)(1-\frac{u^N(t)}{N})u^N(t), \text{ respectively } \gamma_N(t)\frac{(u^N(t))^2}{N}, \tag{3.238}$$

where $\alpha_N(t), \gamma_N(t)$ are defined:

$$\alpha_N(t) = c\int_0^t \sum_{j=2}^{\infty} jU^N(t,ds,j)ds, \quad \gamma_N(t) = c\int_0^t U^N(t,ds,1). \tag{3.239}$$

These rates of change of $U^N(t,\cdot,\cdot)$ follow directly from the dynamics of the dual particle system η.

We note that if we start with k particles at each of ℓ different sites we obtain similar quantities, which we denote by

$$(u_t^{N,k,\ell}, U_t^{N,k,\ell})_{t\geq 0}. \tag{3.240}$$

The dynamics of the the system

$$(u^N(t), U^N(t,ds,dx)) \tag{3.241}$$

evolves in two main regimes corresponding to times $o(\log N)$ and those of the form $\alpha^{-1}\log N + t$, with $t = O(1)$. They are asymptotically as $N\to\infty$ *linear* and *random* for times of order $o(\log N)$ as we have demonstrated in the previous subsubsection with the randomness captured by the random variable W given in (3.143) respectively (3.230) above. On the other hand the dynamics become *deterministic* and *nonlinear* in the post-emergence regime, that is, for times of the form $(T_N+t)_{t\in(-\infty,\infty)}$ where $T_N = \frac{\log N}{\alpha}$. This latter case involves the law of large numbers in the limit $N\to\infty$ and collisions play a decisive role. We will introduce the limiting dynamics in Step 2. We will later in Subsubsection 3.2.9 show how the two regimes are linked and that the first produces a random initial condition for the second, the nonlinear regime.

Step 2 *The limiting dynamics for dual in collision regime*

The second step is to define for $k,\ell\in\mathbb{N}$ candidates

$$(\Pi_t^{k,\ell}, u^{k,\ell}(t), U^{k,\ell}(t,ds,dx))_{t\in\mathbb{R}} \text{ and } (\zeta(t))_{t\in\mathbb{R}} \tag{3.242}$$

for the limiting objects as $N\to\infty$ related to (here $T_N = \alpha^{-1}\log N$)

$$\begin{aligned}&\left(\frac{\Pi^{N,k,\ell}_{T_N+t}}{N}, \frac{u^{N,k,\ell}_{T_N+t}}{N}, U^{N,k,\ell}(T_N+t,ds,dx)\right), \; -\infty<t<\infty,\\ &\{\zeta^{N,(k,\ell)}(T_N+t,i), \quad i=1,\cdots,N\}, \quad -\infty<t<\infty,\end{aligned} \tag{3.243}$$

when the dual process initially has k *particles* at each of ℓ *distinct sites* at time $t=0$.

We now turn to the limiting objects as $N\to\infty$ which we denote

$$(\Pi_t^{k,\ell}, u(t), U(t))_{t\in\mathbb{R}}, \quad (\zeta^{k,\ell}(t))_{t\in\mathbb{R}}. \tag{3.244}$$

We remark that it will turn out that conditioned on $(\Pi_{t_0}^{k,\ell}, u^{k,\ell}(t_0), U^{k,\ell}(t_0,ds,dx))$, the dynamics of $(\Pi_t^{k,\ell}, u^{k,\ell}(t), U^{k,\ell}(t,ds,dx))_{t\geq t_0}$ is independent of k,ℓ. For this reason we often suppress the superscripts and later indicate how the initial conditions for $k,\ell>1$ are determined.

Since

$$\Pi_t^{k,\ell} = u^{k,\ell}(t)\cdot\sum_{j=1}^{\infty} j\cdot U^{k,\ell}(t,\mathbb{R}^+,j), \tag{3.245}$$

and since the process ζ will depend on this pair from the triple in (3.242) it suffices to specify the pair $(u(t), U(t))_{t\in\mathbb{R}}$. This is done by first establishing that they satisfy a pair of deterministic equations and then determine the entrance law at $-\infty$ as follows.

We first define two ingredients (parallel to (3.239)):

$$\alpha(t) = \alpha(U(t)) := c \int_0^\infty \sum_{j=2}^\infty jU(t, ds, j), \quad \gamma(t) = \gamma(U(t)) := c \int_0^\infty U(t, ds, 1). \tag{3.246}$$

As before we first consider the evolution equation for times $t \geq t_0$ and then we try to characterize an entrance law by considering $t_0 \to -\infty$. For the moment, for notational convenience let $t_0 = 0$.

Then to specify

$$(u, U) = (u(t), U(t))_{t\geq 0}, \text{ with } (u(t), U(t)) \in \mathbb{R}_+ \times \mathcal{M}_1(\mathbb{R}_+ \times \mathbb{N}), \tag{3.247}$$

we introduce the (coupled) system of nonlinear evolution equations:

$$\frac{du(t)}{dt} = \alpha(t)(1 - u(t))u(t) - \gamma(t)u^2(t), \tag{3.248}$$

$$\begin{aligned} \frac{\partial U(t, dv, j)}{\partial t} & \\ = \quad & -\frac{\partial U(t, dv, j)}{\partial v} \\ & +s(j-1)1_{j\neq 1}U(t, dv, j-1) - sjU(t, dv, j) \\ & +\frac{d}{2}(j+1)jU(t, dv, j+1) - \frac{d}{2}j(j-1))1_{j\neq 1}U(t, dv, j) \\ & +c(j+1)U(t, dv, j+1) - cjU(t, dv, j)1_{j\neq 1} \\ & -cu(t)U(t, dv, 1)1_{j=1} \\ & +u(t)(\alpha(t) + \gamma(t))[1_{j\neq 1}U(t, dv, j-1) - U(t, dv, j]) \\ & + ((1 - u(t))\alpha(t))\, 1_{j=1} \cdot \delta_0(dv) \\ & -\Big(\alpha(t)(1 - u(t)) - \gamma(t)u(t)\Big) \cdot U(t, dv, j). \end{aligned} \tag{3.249}$$

Note that if $U(0, \cdot, \cdot)$ is a probability measure on $\mathbb{R}_+ \times \mathbb{N}$, then this property is preserved under the evolution. (Recall here that α and γ contain the constant c of migration.)

We note that the right side equation (3.249) is only defined in terms of $\mathcal{M}_{\text{fin}}(\mathbb{R}^+ \times \mathbb{N})$ elements, if we restrict to measures for which $\int_0^\infty \sum_j j^2 \cdot U(dv, j) < \infty$, so we have to restrict to a subset of $\mathcal{M}_1(\mathbb{R}^+ \times \mathbb{N})$. Define ν as the measure on $\mathbb{N}$ given by

$$\nu(j) = 1 + j^2. \tag{3.250}$$

and let $L^1(\nu, \mathbb{N})$ denote the corresponding space of integrable functions.

We can write these equations in compact form as

$$\frac{d}{dt}(u(t), U(t, \cdot, \cdot)) = \vec{G}^*(u(t), U(t, \cdot, \cdot)), \tag{3.251}$$

with the *nonlinear* operator

$$\vec{G}^* \text{ mapping } \mathbb{R}_+ \otimes L^1_+(\mathbb{R}_+ \times \mathbb{N}, \nu) \to \mathbb{R}_+ \otimes \mathbb{F}, \tag{3.252}$$

with $\mathbb{F}$ denoting the class of finite measurable functions f on $\mathbb{R}_+ \times \mathbb{N}$.

If $u(0) = u_0 > 0$, we also work with the equivalent $\mathcal{M}_{\text{fin}}(\mathbb{R}_+ \times \mathbb{N})$-valued process

$$\widehat{\Psi}_t := u(t) \cdot U(t) \tag{3.253}$$

which solves the nonlinear systems

$$\frac{d\widehat{\Psi}_t}{dt} = G^*(\widehat{\Psi}_t) \tag{3.254}$$

equivalent to (3.251) (note that it is easy to verify that if $u(t_0) = \widehat{\Psi}_{t_0}(\mathbb{R}_+ \times \mathbb{N}) > 0$, then $u(t) > 0$ for all $t \geq t_0$).

We think of this as a forward equation for a deterministic measure-valued process. In order to formulate the corresponding Markov semigroup generator we consider functions $F_{g,f}$ on $(\mathbb{R}_+ \otimes L^1_+(\mathbb{R}_+ \times \mathbb{N}, \nu))$ of the form

$$\begin{aligned} F_{g,f}(\widehat{\Psi}_0) &= g\left(\sum_{j=1}^{\infty} \int_0^{\infty} f(r,j)\widehat{\Psi}_0(dr,j)\right) \\ &g : \mathbb{R} \to \mathbb{R},\ f : [0,\infty) \times \mathbb{N} \to \mathbb{R},\ g \text{ has bounded first and second derivatives,} \\ &f \text{ bounded and has bounded first derivatives in the first variable.} \end{aligned} \tag{3.255}$$

The algebra of functions, which contains all functions of the form (3.255) is denoted

$$D_0(\mathbf{G}) \tag{3.256}$$

and is a separating set in $C_b(\mathcal{M}_{\text{fin}}([0,\infty) \times \mathbb{N}_0))$ and is therefore *distribution-determining for laws* on $\mathcal{M}_{\text{fin}}([0,\infty) \times \mathbb{N}_0)$.

Remark 23 *In fact since we are concerned with $\widehat{\Psi}$ which are sub-probability measures we note that the integrals are bounded, we can include unbounded functions g defined on $[0,1]$ and in $C^2([0,1],\mathbb{R})$. In particular, we can consider $g(x) = x$ and $g(x) = x^2$.*

Then we introduce the semigroup

$$V_t F_{g,f}(\widehat{\Psi}_0) := F_{g,f}(\widehat{\Psi}(t)), \tag{3.257}$$

where $\widehat{\Psi}(t) = u(t)U(t)$ and $(u(t), U(t))$ is the solution of the pair (3.248), (3.249) with initial condition $\widehat{\Psi}_0 = u_0 \cdot U_0$. Let

$$\widehat{\Psi}(f) := \left(\sum_{j=1}^{\infty} \int_0^{\infty} f(r,j)\widehat{\Psi}_0(dr,j)\right). \tag{3.258}$$

Then with $F_{g,f}$ as in (3.255) we set

$$
\begin{aligned}
\mathbf{G}F_{g,f}(\widehat{\Psi}) &= g'\left(\widehat{\Psi}(f)\right)\left(\widehat{\Psi}(G^{\widehat{\Psi}}f(r,j))\right) \\
&= g'\left(\widehat{\Psi}(f)\right)\int_0^\infty\int_1^\infty \tfrac{\partial f(r,x)}{\partial r}\widehat{\Psi}(dr,dx) \\
&\quad + g'(\widehat{\Psi}(f))\int_0^\infty\int_1^\infty sx\left[(f(r,x+1)-f(r,x)))\right]\widehat{\Psi}(dr,dx) \\
&\quad + g'(\widehat{\Psi}(f))\int_0^\infty\int_1^\infty \tfrac{d}{2}x(x-1)\left[f(r,x-1)-f(r,x))\right]\widehat{\Psi}(dr,dx) \\
&\quad + g'(\widehat{\Psi}(f))\cdot c(1-\widehat{\Psi}(1))\left(\int_0^\infty\int_1^\infty[f(0,1)+f(r,x-1)-f(r,x)]x1_{x\neq 1}\widehat{\Psi}(dr,dx)\right) \\
&\quad + g'(\widehat{\Psi}(f))\widehat{\Psi}(1)c\int_0^\infty\int_1^\infty\left[f(r,x+1)-f(r,x))\right]x\widehat{\Psi}(dr,dx) \\
&\quad + g'(\widehat{\Psi}(f))c(\int_0^\infty\int_1^\infty x\widehat{\Psi}(dr,dx))\left(\int_0^\infty\int_1^\infty\left[1_{\widetilde{x}\neq 1}f(\widetilde{r},\widetilde{x}-1)-f(\widetilde{r},\widetilde{x}))\right]\widehat{\Psi}(d\widetilde{r},d\widetilde{x})\right) \\
&= g'(\widehat{\Psi}(f))\cdot\int_0^\infty\int_1^\infty G^{\widehat{\Psi}}f(r,x)\widehat{\Psi}(dr,dx),
\end{aligned}
\tag{3.259}
$$

with the obvious extension to $D_0(\mathbf{G})$.

We can then consider the martingale problem associated with $(\mathbf{G}, D_0(\mathbf{G}))$, that is, to determine probability measures, P, on $C([0,\infty),\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B})$ (where $\mathbb{B}$ is defined below in (3.262)) such that

$$
F_{g,f}(\widehat{\Psi}(t)) - \int_0^t(\mathbf{G}F_{g,f})(\widehat{\Psi}(s))ds \tag{3.260}
$$

is a P-martingale for all $F\in D_0(\mathbf{G})$.

Since $\mathbf{G}$ is a *first order* operator any solution to this martingale problem is deterministic. To verify this consider $g(x)=x^2$ and $g(x)=x$ and show that $\mathbf{G}(F_{x^2})-2F_x\mathbf{G}F_x=0$. This holds by inspection of (3.259). This implies that the variance of the marginal distribution $\widehat{\Psi}_t(f)$ for a solution to the martingale problem is 0.

Then taking next $g(x)=x$ we have that any solution satisfies

$$
\widehat{\Psi}_t(f) - \int_0^t\widehat{\Psi}_s(G^{\widehat{\Psi}(t)}f)ds = \widehat{\Psi}_t(f) - \int_0^t G^*\widehat{\Psi}_s(f)ds. \tag{3.261}
$$

Taking $f\equiv 1$ and then indicator functions $f=1_j$ we obtain our nonlinear system. In other words any solution to the martingale problem is a deterministic trajectory satisfying the nonlinear dynamics (3.248), (3.249). This form will play a role later on when we consider the convergence.

In order to discuss the existence and uniqueness of solutions to these equations we introduce a norm and reformulate the equations as a nonlinear evolution equation in a Banach space, namely,

$$
\widehat{\mathbb{B}} := \mathbb{R}\otimes L^1(\mathbb{R}_+\times\mathbb{N}, ds\times\nu) \tag{3.262}
$$

which we furnish with the norm

$$
\|\widehat{\Psi}\| = \|(u,(a_i(\cdot))_{i\in\mathbb{N}})\| = |u| + \|(a_i(\cdot))_{i\in\mathbb{N}}\|_1, \tag{3.263}
$$

where $u=\widehat{\Psi}(\mathbb{R}\times\mathbb{N}),\ a_i(\cdot)=\widehat{\Psi}(\cdot\times\{i\})$ and

$$
\|(a_i(\cdot))_{i\in\mathbb{N}}\|_1 := \sum_{j=1}^\infty(1+j^2)|a_j(\cdot)|_{\text{var}}, \tag{3.264}
$$

where var denotes the total variation norm for measures on $\mathbb{R}_+$. On this space the r.h.s. of the equation (3.249) is well-defined.

We note that the Markov property is satisfied by the equivalent systems $(u(t), U(t, \mathbb{R}, \cdot))$ and $\widehat{\Psi}(t) = u(t) \cdot U(t, \mathbb{R}, \cdot)$. Since this all that is needed for the dual representation we often suppress the ages. In this case we use

$$\mathbb{B} := \mathbb{R} \otimes L^1(\mathbb{N}, \nu) \tag{3.265}$$

and the simpler norm

$$\|(u, (a_i))_{i\in\mathbb{N}})\| = |u| + \|(a_i)_{i\in\mathbb{N}}\|_1, \quad \text{where} \quad \|(a_i)_{i\in\mathbb{N}}\|_1 := \sum_{j=1}^{\infty}(1+j^2)|a_j|. \tag{3.266}$$

Existence of a solution to this system will follow from the convergence result below in Subsubsection 3.2.8 (Proposition 3.8). The question of uniqueness given the initial point $(u(t_0), U(t_0))$ is considered in Step 3 below.

Return now to the finite N-system and imagine the collection of occupied sites in our dual particle process at a specific time t represented in the form $\zeta^N(t, i), i = 1, \ldots, N_t$. To complete the description we must specify what happens at a single tagged site for times $r \geq t$, i.e. to $\zeta^N(r, i), i \in \mathbb{N}$ in the limit $N \to \infty$.

Given the function $(u(t), U(t))$ this will be given by the time-inhomogeneous birth, death and immigration process

$$\{\zeta(t, i), t \geq 0\}, \text{ with transition probabilities } \{p_{k,j}(t), (k, j) \in \mathbb{N}^2, \quad t \geq 0\}, \tag{3.267}$$

given by the *forward* nonlinear Kolmogorov equation starting with one particle at site i at time 0, i.e., $p_{1,1}(0) = 1$ and which reads as follows:

$$p'_{1,j}(t) = (s(j-1) + \alpha(t)u(t))p_{1,j-1}(t) + \frac{d}{2}(j+1)jp_{1,j+1}(t) - ((s+c)j + \frac{d}{2}j(j-1))p_{1,j}(t) \tag{3.268}$$

for $j \neq 0, 1$,

$$p'_{1,0}(t) = cu(t)p_{1,1}(t) - \alpha(t)u(t)p_{1,0}(t), \tag{3.269}$$

$$p'_{1,1}(t) = d \cdot p_{1,2}(t) - ((s + \alpha(t)u(t)))p_{1,1}(t), \tag{3.270}$$

where

$$u(t) = 1 - p_{1,0}(t), \quad \alpha(t)u(t) = c\sum_{j=2}^{\infty} jp_{1,j}(t). \tag{3.271}$$

This process $\zeta(t, i)$ will describe the evolution of a tagged site i in the limiting system in the collision regime.

Next, in order to prepare for the study of the nonlinear system (3.248),(3.249) we consider a related set of equations for the non-collision regime obtained by setting $u \equiv 0$, namely,

$$\begin{aligned}\frac{\partial\widetilde{U}(t,dv,j)}{\partial t} = \quad & -\frac{\partial\widetilde{U}(t,dv,j)}{\partial v} \\ & + s(j-1)1_{j\neq 1}\widetilde{U}(t,dv,j-1) - sjU(t,j) \\ & + \frac{d}{2}(j+1)j\widetilde{U}(t,dv,j+1) - \frac{d}{2}j(j-1))1_{j\neq 1}\widetilde{U}(t,dv,j) \\ & + c(j+1)\widetilde{U}(t,dv,j+1) - cj\widetilde{U}(t,dv,j)1_{j\neq 1} \\ & + \widetilde{\alpha}(t)[\delta_0 \times 1_{j=1} - \widetilde{U}(t,dv,j)],\end{aligned} \tag{3.272}$$

where $\widetilde{\alpha}(t) := c\sum_{\ell=2}^{\infty}\int_0^{\infty} \ell \cdot \widetilde{U}(t,dv,\ell)$.

Since the dual representation expressions only involve the marginals $U(t,\mathbb{R}_+,j)$ and the jump rates depend on j but not the age of an occupied site it suffices to work with these marginal distributions we denote by $U(t,j)$. We then work with the corresponding versions of (3.248),(3.249) and (3.272) and use the norm (3.266).

We observe next that the solution to the nonlinear system (3.272) can be obtained as

$$\widetilde{U}(t,dv,j) = \frac{V(t,dv,j)}{\sum_j \int_0^{\infty} V(t,dv,j)} \tag{3.273}$$

where $\{V(t,\cdot,\cdot)\}$ solve the *linear* system which we spell out explicitly later on, namely it is obtained by deleting the last term in (3.289). The resulting linear equations coincide with the mean equations for the CMJ process and hence the convergence of $V(t,\cdot,\cdot)$ as $t\to\infty$ to the stable size and age distribution follows from the CMJ theory. In addition we have that $\alpha(t)\to\alpha$ as $t\to\infty$, the Malthusian parameter of the CMJ process.

We define a linear operator given by the infinite matrix

$$Q^{a,*} \text{ on } L^1_+(\mathbb{N},\nu) \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}), \tag{3.274}$$

obtained by first integrating over ages setting $\alpha(t)=a$ on the right side of the (3.272). Note that the equation resulting this way from (3.272) can be viewed as the forward equations for a Markov chain Z on $\mathbb{N}$ with transition matrix $Q^{a,*}$. Let

$$(S^a)_{t\geq 0} = \text{ the dual semigroup on } \mathcal{M}_1(\mathbb{N}) \text{ of this time-homogeneous Markov chain on } \mathbb{N}. \tag{3.275}$$

To verify that the semigroup S^α_t corresponding to $Q^{\alpha,*}$ is strongly continuous on $(\mathbb{B},\|\cdot\|)$, note that for $\mu\in\mathcal{M}_1(\mathbb{N})$,

$$\lim_{t\to 0}\|S_t\mu-\mu\| = \lim_{t\to 0}[E[Z^2(t)]-E[Z^2(0)]] = 0, \tag{3.276}$$

follows from elementary properties of the Markov chain $(Z(t))_{t\geq 0}$.

Note that the generator of S^α is an unbounded operator but has domain containing

$$D_0 := \{(a_i)_{i\in\mathbb{N}} \in \mathbb{B} : \sum_{j=1}^{\infty}(1+j^4)|a_i| < \infty\} \tag{3.277}$$

and in fact this is a core for the semigroup ([EK2], Ch. 1, Prop. 3.3), since the birth and death process with linear birth and quadratic death rates has moments of all orders at positive times. (Note that the corresponding Markov process on $\mathbb{N}$ has an entrance law starting at $+\infty$. This property is wellknown for the Kingman coalescent but extends to our birth and death process provided $d>0$ and is easily extended to include an additional immigration term.)

The Markov chain associated with $Q^{a,*}$ has a unique equilibrium state $U^a(\infty)$. Let $m(a)$ denote the mean equilibrium value

$$m(a) = \sum_{j=2}^{\infty} j\cdot U^a(\infty,j). \tag{3.278}$$

The equation $m(a)=a$ has a unique positive solution and it is equal to α. Moreover S^α_t satisfies

$$\lim_{t\to\infty}(S^\alpha_t U(0)) = \mathcal{U}(\infty), \quad \limsup_{t\to\infty}|\alpha(t)-\alpha| = 0, \tag{3.279}$$

where $\mathcal{U}(\infty) = U^\alpha(\infty)$ is the unique equilibrium with mean α and is the stable size distribution for the CMJ process.

Step 3 *Uniqueness and existence results*

We next show that the equations we gave in Step 2 uniquely determine the objects needed for our convergence results.

Lemma 3.6 *(Uniqueness of the pair (u, U) and of ζ)*

(a) The pair of equations (3.248) - (3.249), for $(u(t), U(t, \mathbb{R}_+, \cdot))$ given an initial state from $\mathbb{R}^+ \times \mathcal{M}_1(\mathbb{N})$ satisfying

$$(u(t_0), U(t_0, \cdot)) \in [0,1] \otimes \mathcal{M}_1(\mathbb{N}) \cap L^1(\mathbb{N}, \nu) \tag{3.280}$$

at time t_0, has a unique solution $(u(t), U(t))_{t \geq t_0}$ with values in $[0,1] \otimes (\mathcal{M}_1(\mathbb{N}) \cap L^1_+(\mathbb{N}, \nu))$.

(b) Given (u, U) the process $(\zeta(t))_{t \geq t_0}$ is uniquely determined by (3.268).

(c) There exists a solution (u, U) with time parameter $t \in \mathbb{R}$ for every $A \in (0, \infty)$ with values in $\mathbb{B} = \mathbb{R} \otimes L^1(\mathbb{N}, \nu)$, such that

$$u(t) e^{-\alpha t} \to A \text{ as } t \to -\infty, \tag{3.281}$$

$$U(t) \underset{t \to -\infty}{\longrightarrow} \mathcal{U}(\infty), \quad \text{and } \limsup_{t \to -\infty} e^{-\alpha t} |\alpha(t) - \alpha| = B < \infty. \tag{3.282}$$

Here $\mathcal{U}(\infty)$ is the stable age and size distribution of the CMJ-process corresponding to the particle process $(K_t, \zeta_t)_{t \geq 0}$ given by the McKean-Vlasov dual process η, which was defined in (3.166).

(d) Given any solution (u, U) of equations (3.248) - (3.249) integrating out the age (given in (3.287)-(3.288)) for $t \in \mathbb{R}$ with values in the space $\mathbb{B} := \mathbb{R} \otimes L^1(\mathbb{N}, \nu)$ satisfying

$$u(t) \geq 0, \quad \limsup_{t \to -\infty} e^{-\alpha t} u(t) < \infty, \quad t \to \|U(t)\| \text{ is bounded,} \tag{3.283}$$

has the following property of u

$$A = \lim_{t \to -\infty} e^{-\alpha t} u(t) \tag{3.284}$$

exists and the solution of (3.248)-(3.249) satisfying (3.284) for given A is unique. Furthermore $U(t)$ and $\alpha(t)$ satisfy

$$U(t) \Longrightarrow \mathcal{U}(\infty) \text{ as } t \to -\infty, \qquad \text{and } \limsup_{t \to -\infty} e^{-\alpha t} |\alpha(t) - \alpha| < \infty. \tag{3.285}$$

(e) The solution of the system including the age distribution, that is, (3.248) and (3.249) has a unique solution. $\square$

The following is immediate from (3.280) and the discussion in (3.272)-(3.261).

Corollary 3.7 *The $(\mathbf{G}, D_0(\mathbf{G}))$ martingale problem is well-posed.*

Remark 24 *Note that looking at the form of the equation we see that a solution indexed by $\mathbb{R}$ remains a solution if we make a time shift. This corresponds to the different possible values for the growth constant A in (3.281). In particular the entrance law from 0 at time $-\infty$ is unique up to the time shift.*

Proof of Lemma 3.6

(a) First define

$$b(t) = 1 + \frac{\gamma(t)}{\alpha(t)} \tag{3.286}$$

and we express $\alpha(t)$ and $\gamma(t)$ in terms of $U(t)$ and and consider $L^1(\nu,\mathbb{N})$ as a basic space for the analysis of the component $U(t,\cdot)$. Then we obtain a system of coupled differential equations for the pair (u,U):

$$\frac{du(t)}{dt} = \alpha(t)(1-b(t)u(t))u(t), \tag{3.287}$$

$$\begin{aligned}\frac{\partial U(t,j)}{\partial t} = \quad & +s(j-1)1_{j\neq 1}U(t,j-1) - sjU(t,j)\\ & +\frac{d}{2}(j+1)jU(t,j+1) - \frac{d}{2}j(j-1))1_{j\neq 1}U(t,j)\\ & +c(j+1)U(t,j+1) - cjU(t,j)1_{j\neq 1}\\ & +\left[\alpha(t)-\alpha(t)u(t)-\gamma(t)u(t))\right]1_{j=1}\\ & +u(t)(\alpha(t)+\gamma(t))[U(t,j-1)1_{j\neq 1} - U(t,j)]\\ & -\Big(\alpha(t)(1-u(t)) - \gamma(t)u(t)\Big)\cdot U(t,j).\end{aligned} \tag{3.288}$$

We now consider the related nonlinear system obtained by setting $u\equiv 0$ in (3.288), namely,

$$\begin{aligned}\frac{\partial \widetilde{U}(t,j)}{\partial t} = \quad & +s(j-1)1_{j\neq 1}\widetilde{U}(t,j-1) - sjU(t,j)\\ & +\frac{d}{2}(j+1)j\widetilde{U}(t,j+1) - \frac{d}{2}j(j-1))1_{j\neq 1}\widetilde{U}(t,j)\\ & +c(j+1)\widetilde{U}(t,j+1) - cj\widetilde{U}(t,j)1_{j\neq 1}\\ & +\alpha(t)1_{j=1} - \alpha(t)\cdot\widetilde{U}(t,j).\end{aligned}$$

where $\alpha(t) = c\sum_{\ell=2}^{\infty}\ell\cdot\widetilde{U}(t,\ell)$.

We will consider the system (3.288) as a nonlinear perturbation of the linear strongly continuous semigroup S^α on the Banach space $\mathbb{B} = L^1(\nu,\mathbb{N})$ given in (3.275).

Then we can rewrite (3.287) and (3.288) in the form (G^* stands for adjoint of G)

$$\frac{d}{dt}(u(t),U(t,\cdot)) = \vec{G}^*(u(t),U(t,\cdot)) = (0,U(t,\cdot)Q^{\alpha,*}) + F(u(t),U(t,\cdot)), \tag{3.289}$$

where the infinite matrix $Q^{\alpha,*}$ is the infinitesimal generator (Q-matrix) of the Markov semigroup of S^α and denote by $U^{\alpha,*}(t)$ the state if S^α acts on measures and $F:\mathbb{B}\longrightarrow\mathbb{B}$ is a (locally) Lipschitz continuous function on $\mathbb{B}$, namely,

$$\begin{aligned}F(u,U) = \big(\alpha(U)u - (\gamma(U)+\alpha(U))u^2,\ ((\alpha-\alpha(U)) + u(\alpha(U)+\gamma(U))))Z(U)\big)\,,\\ \text{where } Z(U) = (Z_j)_{j\in\mathbb{N}},\ \text{with } Z_j = U(j-1), j\neq 1, Z_1 = -1.\end{aligned} \tag{3.290}$$

Note that F involves a third degree polynomial in u and $\alpha(t)$. Moreover $U\to\alpha(U)$ is a linear function and differentiable on $\mathbb{B}$. Therefore we can conclude that F is C^2 with first and second derivatives uniformly bounded on bounded subsets of $\mathbb{B}$. Therefore we can use the result of Marsden [MA], Theorem 4.17 (see Appendix 4) to conclude that there is a unique flow $(u(t),U(t))$ satisfying the equations and in addition the mapping $t\to(u(t),U(t))$ and the mapping $(u_{t_0},U_{t_0})\to(u(t),U(t))$ are Lipschitz in $\mathbb{B}$.

Remark 25 (Alternate proof of (a))

We write the equation (3.244) as an a system of ODE in the form

$$B'(t) = \vec{G}^*(B(t)) = L(B(t)) + N(B(t)), \tag{3.291}$$

where L is a linear operator (generator of a Markov process) and N is a nonlinear operator on the Banach space $\mathbb{B}$. The latter is locally Lipschitz but not globally Lipschitz. Furthermore the operator L maps $\mathbb{B}$ into finite well-defined functions on $\mathbb{N}$, but not *into $\mathbb{B}$. Only if we restrict to the smaller sets of configurations $\widetilde{\mathbb{B}}$, i.e. configurations b satisfying $b \in L^1(\mathbb{N}, \nu)$ and $\sum j^4 |(b(j)| < \infty$ we obtain $Lb \in \mathbb{B}$. This subset of configurations forms a dense subset in $L^1(\mathbb{N}, \nu)$. This means we cannot view this as a standard evolution equation in the Banach space $\mathbb{B}$ and have to bring in some extra information to make the usual arguments work. Namely we have to show that if we start in the set $\widetilde{\mathbb{B}}$ we remain there; we get an evolution in the Banach space $\mathbb{B}$. The extra information needed comes from the probabilistic interpretation.*

Indeed the evolution equation

$$\widetilde{B}'(t) = Q^*(\widetilde{B}(t)) \tag{3.292}$$

has a nice dual Markov semigroup (backward equation) with which we can work. Namely we know that the corresponding Markov process on $\mathbb{N}$ has an entrance law starting at $+\infty$. This property is wellknown for the Kingman coalescent but extends to our birth and death process provided $d > 0$. This is easily extended furthermore to an additional immigration term.

Hence the evolution (forward equation) can be started with "$\widetilde{B}(0) = \delta_\infty$" and for all solution we obtain a smaller path in stochastic order, in fact the entrance law from infinity has the property that in particular $\sum_j \widetilde{B}(t,j)(1+j^4) < \infty$, for $t > 0$. Therefore we can solve the evolution equation (3.292) in $\mathbb{B}$ uniquely. This has to be extended to the nonlinear equations where again starting somewhere in $\mathbb{B}$ after any positive time we reach the stronger summability condition.

However if we truncate L to L^k where $L^k x = 1_{\{j \le k\}} Lx$, we can view (3.244) as a nonlinear evolution equation in a subset of the Banach space $\mathbb{R} \otimes L^1(\mathbb{N}, \nu)$ (recall (3.263)). This means now we are in the set-up of ODE's in Banach spaces *and can use the classical theory of existence and uniqueness for such equations to obtain unique solutions B^k.*

Then we return to the original equation observing that for the solution B^k of the truncated problem we have for initial states with $\sum_i B(0,i)(i)^4 < \infty$ as $k \to \infty$

$$(B^k(t))' \longrightarrow B'(t), \quad B^k(t) \longrightarrow B(t), \quad t \ge 0, \tag{3.293}$$

using the bound in stochastic order from the entrance law as pointed out above.

If we have existence of the solution in $R^+ \otimes L^1_+(\mathbb{N}, \nu)$ for all t (where $L^1_+(\mathbb{N}, \nu)$ denotes the set of sequences in $L^1(\mathbb{N}, \nu)$ having non-negative components) it suffices to show local uniqueness.

Turn first to the existence problem. We shall obtain in Proposition 3.8 that we have convergence of the finite N systems to a solution of the equations and we need for the existence to know that it is in our Banach space. It can easily be verified that the solution obtained in the limit must lie in $\mathbb{R}^+ \times L^1_+(\mathbb{N}, \nu)$ and has no explosions using apriori estimates. (Recall that for $\zeta^N(t,i)$ the second moments (in fact all *moments) are finite over finite time intervals since this is true for the process without collision which provides upper bounds on u^N, U^N independently of N.) Using the fact that second moments remain bounded in bounded intervals the proof is standard. Therefore a solution arising as a limit for $N \to \infty$ lies in the subspace of $\mathbb{R}^+ \times \mathcal{M}_1(\mathbb{N})$ where $\| \cdot \|$ is finite. This gives existence of a solution with values in the specified Banach space.*

We turn to the uniqueness. The form of the equation now immediately gives the local Lipschitz property of F. Therefore the solution is locally unique by the uniqueness result for nonlinear differential equations in Banach spaces (e.g. [Paz], Chapt. 6, Theorem 1.2) applied to the truncated problem and together with the approximation property we get uniqueness of our equation.

(b) It is standard to check that ζ is for given (u, U) and for (3.268)-(3.270) a nonexplosive time-inhomogeneous Markov process which is uniquely determined by these equations. We have to show that the self-consistency relation (3.271) can be satisfied. This will follow from our convergence result for (u^N, U^N) below in Proposition 3.8, Part (e).

(c) To prove existence of an entrance law with the prescribed properties, we construct a solution as a limit of the sequence $(u_n(t), U_n(t))_{t\in\mathbb{R}}, n \in \mathbb{N}$ defined as follows:

$$\begin{aligned} u_n(t) &= Ae^{\alpha t_n} \text{ for } t \le t_n \\ U_n(t) &= \mathcal{U}(\infty) \text{ for } t \le t_n \\ (u_n, U_n) &\quad \text{satisfy (3.287) and (3.288) for } t \ge t_n. \end{aligned} \tag{3.294}$$

Standard arguments imply the tightness of the sequence $\{(u_n(t), U_n(t))_{t\in\mathbb{R}}, n \in \mathbb{N}\}$ and that any limit point (u, U) satisfies (3.287) and (3.288). We now have to show that the limit points satisfy the conditions on the behaviour as $t \to -\infty$.

We observe that the solution to the equation for $u \equiv 0, U \equiv \mathcal{U}(\infty)$ gives a stationary solution of the coupled pairs of equations. We know that the growth by $Ae^{\alpha t}, t \in \mathbb{R}$ is an upper bound for the u-part of the limit, since this is the collision-free solution of the equation and therefore $\limsup\limits_{t\to\infty}(e^{-\beta t}u(t)) \le A < \infty$. We therefore must verify that the limit point matches exactly that growth and starts from $\mathcal{U}(\infty)$ at time $t = -\infty$. We note that (u, U) must then agree with a stationary solution for $u \equiv 0$, which is unique by standard Markov chain theory and hence the point at $t = -\infty$ must be $\mathcal{U}(\infty)$. We next argue that a limit point (u, U) actually satisfies $e^{-\beta t}u(t)$ converges to A as $t \to -\infty$. We shall see below that this would actually follow if $u(t)e^{-\beta t}$ has for $t \to -\infty$ A as limes superior. If we know that the solution with this property is unique, then our limit point in the construction must then have the desired properties and in particular we must actually have convergence of (u_n, U_n) as $u \to \infty$. This however we shall show below in (d).

(d) Let (u, U) be a solution satisfying (3.283). Then there exists a sequence $t_n \to -\infty$ such that as $n \to \infty$ the limit of $(u(t_n), U(t_n))$ exists.

Now consider the nonlinear system $(u(t), U(t))$. We must verify that as $t \to -\infty$ we actually converge to the case where $u \equiv 0$ and $\alpha(t) \equiv \alpha$. First note that by (3.130) we can assume that $\alpha(t)$ is bounded for all t. By tightness we can find a sequence $(t_n)_{n\in\mathbb{N}}, \quad t_n \to -\infty$ as $n \to \infty$ and $A \in (0, \infty)$ such that as $n \to \infty$, $u(t_n)e^{-\alpha t_n} \to A$. Since by assumption $\limsup_{t\to-\infty} e^{-\alpha t}u(t) < \infty$ it follows from the equations that $|\alpha(t) - \alpha| \le const \cdot e^{\alpha t}$. Then any such solution must satisfy

$$u(t) - Ae^{\alpha t} = o(e^{\alpha t}). \tag{3.295}$$

We next note that given $u(t), \alpha(t), \gamma(t)$, the equation for $U(t)$ is linear and has a unique solution by the standard Markov chain theory. Therefore it suffices to prove that $u(t), \alpha(t), \gamma(t)$ as functions on $\mathbb{R}$ are unique given the asymptotics behaviour of u as $t \to -\infty$. Remember that $\alpha(t)$ and $y(t)$ are functions $U(t)$.

We next consider the leading asymptotic term for U. Let $(S^\alpha(t))_{t\ge0}$ the semigroup defined in (3.275) and $S^{\alpha,*}$ is action on measure, $U^{\alpha,*}$ its path, $U^{\alpha,*}(\infty)$ its equilibrium measure. Furthermore let $L(\cdot) = \alpha(\cdot) + \gamma(\cdot)$. Let $t_n, t_m \to -\infty$ so that $t_n - t_m \to \infty$. Then by (3.244)

$$\begin{aligned} U(t_n) = \lim_{t_m\to-\infty} \big[& S^{\alpha,*}(t_n - t_m)U(t_m) + \int_{t_m}^{t_n} S^{\alpha,*}(t_n - s)(\alpha - \alpha(s))Z(U(s))ds \\ & + \int_{t_m}^{t_n} u(s)\, S^{\alpha,*}(t_n - s)L(s)ds \big] \\ = \lim_{t_m\to-\infty} & \big[S^{\alpha,*}(t_n - t_m)U(t_m)\big] + \int_{-\infty}^{t_n} S^{\alpha,*}(t_n - s)(\alpha - \alpha(s))Z(\mathcal{U}(\infty))ds \end{aligned} \tag{3.296}$$

$$+\int_{-\infty}^{t_n} Ae^{\alpha s}\, S^{\alpha}(t_n-s)(\alpha+\gamma)Z(U^{\alpha,*})ds+o(e^{\alpha t_n})$$
$$=\mathcal{U}(\infty)+\int_{-\infty}^{t_n} S^{\alpha,*}(t_n-s)(\alpha-\alpha(s))Z(\mathcal{U}(\infty))ds$$
$$+\int_{-\infty}^{t_n} Ae^{\alpha s}\, S^{\alpha,*}(t_n-s)(\alpha+\gamma)Z(U^{\alpha,*}(\infty))ds+o(e^{\alpha t_n}).$$

Therefore we have

(3.297) $$\lim_{t_n\to-\infty} U(t_n)=\mathcal{U}(\infty).$$

Note that then also $\alpha(t_n)\to\alpha$ and $\gamma(t_n)\to\gamma$ as $n\to\infty$, since these are continuum functionals of $U(t_n)$ and we have the integrability properties.

Finally, assume that there are two solutions. We shall show below that if two such solutions (u_1,U_1) and (u_2,U_2) were distinct a contradiction would result. Hence in particular in (c) we must have convergence as claimed.

Let

(3.298) $$(v(t),V(t)):=(u_1(t),U_1(t))-(u_2(t),U_2(t)),\ \widehat{\alpha}(t)=\alpha_1(t)-\alpha_2(t),\ \widehat{\gamma}(t)=\gamma_1(t)-\gamma_2(t).$$

¿From the above (i.e. (3.295) and using $(\alpha-\alpha(s))=o(e^{\alpha s})$ in (3.296) and estimating therein the third term by explicit bound) we know that

(3.299) $$|v(t_n)|=o(e^{\alpha t_n})\quad\text{and } \|V(t_n)\|=o(e^{\alpha t_n}).$$

Then

(3.300)
$$\begin{aligned}\frac{dv(t)}{dt}&=\alpha v(t)+e^{\alpha t}\widehat{\alpha}(t)-(\alpha+\gamma)e^{\alpha t}v(t)-e^{2\alpha t}(\widehat{\alpha}+\widehat{\gamma}),\\ \frac{dV(t)}{dt}&=V(t)Q^{\alpha,*}-\alpha V(t)+A(\alpha+\gamma)e^{\alpha t}Z((V(t)))+(\alpha+\gamma)\cdot v(t)Z(U^{\alpha,*})\\ &\quad+e^{\alpha t}((\widehat{\alpha}+\widehat{\gamma}+Z(V(t))),\\ \frac{d\widehat{\alpha}(t)}{dt}&=-\alpha\widehat{\alpha}(t)+A(\alpha+\gamma)e^{\alpha t}\alpha(Z(V(t)))+(\alpha+\gamma)\cdot v(t)\alpha(Z(U^{\alpha,*}))+o(e^{\alpha t}),\end{aligned}$$

where $\widehat{\alpha}(t)=\alpha(V(t))$.

We then obtain by inspection from (3.300) that:

(3.301) $$\frac{d(|v(t)|+\|V(t)\|)}{dt}\le\alpha\cdot(1+\mathrm{const}\cdot\mathrm{e}^{\alpha t})(|\mathrm{v(t)}|+\|\mathrm{V(t)}\|).$$

Then use the v equation and (3.299), more precisely we use the V equation to get an expression for $\widehat{\alpha}(t)$ and $\widehat{\gamma}(t)$ as a linear function of v. to conclude $v(t)\equiv 0$ as follows. Namely we conclude using Gronwall's inequality,

(3.302) $$|v(t_n)|+\|V(t_n)\|\le o(e^{\alpha t_m})e^{\alpha(t_n-t_m)}.$$

Letting $t_m\to-\infty$ we obtain

(3.303) $$|v(t_n)|+\|V(t_n)\|=0.$$

This completes the proof of uniqueness of $(u(t),U(t))$.

(e) It remains to show that not only $U(t,\mathbb{R}^+,\cdot)$ is unique but that this holds also for $U(t,\cdot,\cdot)$. Note that our result already implies that $u(\cdot)$ is uniquely determined. Note also $\alpha(t)$ and $\gamma(t)$

depend only on U integrated over the age and are continuous functions of t and are therefore also uniquely determined. Therefore $u(\cdot), \alpha(\cdot), \gamma(\cdot)$ are uniquely determined. Hence we can insert the unique objects as *external* (time-inhomogeneous) input into the equation. Then the equation (3.249) for given u, α, γ is the forward equation of a time inhomogeneous Markov process with state space $M_1([0,\infty) \times \mathbb{N})$ restricted to elements in $\mathbb{R} \otimes L^1(\mathbb{N}, \nu)$, which has a unique solution by standard arguments.

This completes the proof of Lemma 3.6.

3.2.8 The dual process in the collision regime: convergence results

Now we are ready to state the limit theorem for the growth dynamics of the dual population in the critical time scale where collisions are essential, with the key result for the application to the original process given in e) of Proposition 3.8 below, see in particular (3.320). As usual we denote by $\Longrightarrow$ convergence in law.

Consider the dual process which starts with k particles at each of ℓ distinct sites and denote the corresponding functionals of the dual process by $(\Pi_s^{N,k,\ell}, u_s^{N,k,\ell}, U_s^{N,k,\ell})_{s \geq 0}$. In order to consider them in the time scale in which fixation occurs, we introduce:

$$(3.304) \quad \widetilde{u}^{N,k,\ell}(t) = u^{N,k,\ell}((\frac{\log N}{\alpha} + t) \vee 0),\ t \in (-\infty, T], \quad \widetilde{u}^{N,k,\ell}(-\frac{\log N}{\alpha}) = \ell$$

$$(3.305) \quad \widetilde{U}^{N,k,\ell}(t) = U^{N,k,\ell}((\frac{\log N}{\alpha} + t) \vee 0),\ t \in (-\infty, T], \quad \widetilde{U}^{N,k,\ell}(-\frac{\log N}{\alpha}) = \delta_{(k,0)}.$$

Furthermore we need the standard solution of the nonlinear system (3.248) and (3.249), (see Lemma 3.6 part (c)) denoted

$$(3.306) \quad (u^*(t), U^*(t,\cdot,\cdot)))_{t \geq t_0},$$

which is the solution satisfying (recall $\mathcal{U}(\infty)$ was the stable age and size distribution)

$$(3.307) \quad \lim_{t \to -\infty} e^{-\alpha t} u^*(t) = 1, \quad \lim_{t \to -\infty} U^*(t) = \mathcal{U}(\infty).$$

Finally we have to specify three time scales. Let t_0 stand for some element of $\mathbb{R}$ and let $t_0(N)$ and $s(N)$ be as follows:

$$(3.308) \quad t_0(N) = s(N) - \frac{\log N}{\alpha},$$

with

$$(3.309) \quad s(N) \to \infty, s(N) = o(\log N).$$

Recall the time shift in (3.304) and (3.305).

The behaviour of the dual particle system is asymptotically as $N \to \infty$ as follows.

Proposition 3.8 *(Growth of dual population in the critical time scale in the $N \to \infty$ limit)*

(a) Assume that for some $t_0 \in \mathbb{R}$ as $N \to \infty$, $(\frac{1}{N}\widetilde{u}^{N,k,\ell}(t_0), \widetilde{U}^{N,k,\ell}(t_0))$ converges in law to the pair $(u(t_0), U(t_0))$ automatically contained in $[0,\infty) \times L_1(\mathbb{N}, \nu)$.

Then as $N \to \infty$

$$(3.310) \quad \mathcal{L}\left[(\frac{1}{N}\widetilde{u}^{N,k,\ell}(t),\, \widetilde{U}^{N,k,\ell}(t,\cdot,\cdot)))_{t \geq t_0} \right] \Longrightarrow \mathcal{L}\left[(u(t), U(t,\cdot,\cdot)))_{t \geq t_0} \right],$$

in law on pathspace, where the r.h.s. is supported on the solution of the nonlinear system (3.248) and (3.249) corresponding to the initial state $(u(t_0), U(t_0))$. (Note that the mechanism of the limit

dynamics does not depend on k or ℓ, but the state at t_0 will. That is why we suppress k, ℓ on the r.h.s.)

(b) The collection $\{(\widetilde{u}^{N,k,\ell}, \widetilde{U}^{N,k,\ell}), N \in \mathbb{N}\}$ can be constructed on a common probability space such that for $t_0(N)$ as in (3.308) we have:

$$(e^{-\alpha s(N)}\widetilde{u}^{N,k,\ell}(t_0(N)), \widetilde{U}^{N,k,\ell}(t_0(N))) \to (W^{k,\ell}, \mathcal{U}(\infty)), \text{ a.s., as } N \to \infty. \tag{3.311}$$

(c) The scaled occupation density converges (recall (3.304) and (3.305) and (3.307)):

$$\frac{\widetilde{u}^{N,k,\ell}}{N}(t) \underset{N\to\infty}{\Longrightarrow} u^{k,\ell}(t), \ t \in (-\infty, T], \tag{3.312}$$

in the sense that for $\varepsilon > 0$ (with $t_0(N)$ as in (3.308)),

$$P\left(\sup_{t_0(N)\leq t\leq T} e^{-\alpha t}|\frac{\widetilde{u}^{N,k,\ell}}{N}(t) - u^{k,\ell}(t)| > \varepsilon\right) \to 0 \text{ as } N \to \infty. \tag{3.313}$$

Also the age-size distribution of the dual converges:

$$\widetilde{U}^{N,k,\ell}(t,\cdot,\cdot) \underset{N\to\infty}{\Longrightarrow} U^{k,\ell}(t,\cdot,\cdot), \ t \in (-\infty, T], \tag{3.314}$$

in the sense that for $\eta > 0$, with $\|\cdot\|$ denoting the variational norm,

$$\lim_{N\to\infty} P\left(\sup_{t_0(N)\leq t\leq T} \|\widetilde{U}^{N,k,\ell}(t,\cdot,\cdot) - U^{k,\ell}(t,\cdot,\cdot)\| > \eta\right) = 0. \tag{3.315}$$

(d) The limits $(u^{k,\ell}(t), U^{k,\ell}(t,\cdot,\cdot)))_{t\geq t_0}$, can be represented as the unique solution of the non-linear system (3.248) and (3.249) satisfying

$$\lim_{t\to-\infty} e^{-\alpha t}u^{k,\ell}(t) = W^{k,\ell}, \ \lim_{t\to-\infty} U^{k,\ell}(t) = \mathcal{U}(\infty), \tag{3.316}$$

with $W^{k,\ell}$ having the law of the random variable appearing as the scaling (by $e^{-\alpha t}$) limit for $t \to \infty$ of the CMJ-process $K_t^{k,\ell}$ started with k particles at each of ℓ sites.

These solutions are random time shifts of the standard solution (3.306), (3.307), namely:

$$u^{k,\ell}(t) = u^*(t + \frac{\log W^{k,\ell}}{\alpha}), \quad U^{k,\ell}(t) = U^*(t + \frac{\log W^{k,\ell}}{\alpha}). \tag{3.317}$$

(e) Let $(\Pi_u^{N,k,\ell})_{u\geq 0}$ denote the number of dual particles and $T_N = \alpha^{-1}logN$ and $t_0(N), s(N)$ as in (3.308). Then for each $k \in \mathbb{N}, \ell \in \mathbb{N}$ we have:

$$\mathcal{L}[\{N^{-1}\Pi_{T_N+t}^{N,k,\ell}, \ t_0(N) < t < \infty\}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\{\frac{u^{k,\ell}(t)(\alpha(t)+\gamma(t))}{c}\}_{t\in\mathbb{R}}], \tag{3.318}$$

$$\mathcal{L}[(N^{-1}\Pi_u^{N,k,\ell} \quad , \quad 0 \leq u \leq s(N))] \underset{N\to\infty}{\Longrightarrow} \delta_{\underline{0}}, \tag{3.319}$$

with $\underline{0}$ the function on $\mathbb{R}^+$, which is identically zero.

Furthermore for every $t \in \mathbb{R}$, there exists (a deterministic) $\nu_{k,\ell}(t) \in \mathcal{P}([0,\infty))$ such that

$$\mathcal{L}[\frac{1}{N}\int_0^{T_N+t} \Pi_u^{N,k,\ell} du] \underset{N\to\infty}{\Longrightarrow} \nu_{k,\ell}(t). \qquad \square \tag{3.320}$$

Remark 26 *Note that the dual at time T describes the genealogy of a typical sample from the original population drawn at the time T. Here time runs backwards, in particular the evolution of the dual with total time horizon $T = \alpha^{-1}\log N + t$ in the time described by varying t corresponds in the genealogy of the tagged sample to the early moments, while the emergence phenomena of the dual population (arising collisions) reflect in the original population later times, where law of large number effects rule the further expansion of the fitter type on macroscopic scale (i.e. observing the complete space).*

In the case in which the rare mutant has succeeded our result tells us that we have a situation where the mutations which occur very early in the typical ancestral path have generated $O(N)$ possible choices for a fitter type to occur and to then prevail at the observation time at a fixed observation site. The randomness enters since these very early mutations during the first moments of the evolution have in the limit $N \to \infty$ a Poisson structure. This produces a random time shift of emergence and takeover which then follows a deterministic track. This is the global picture, if we look at it locally we find that also a collection of tagged sites follows a random evolution in small time scale.

Proof of Proposition 3.8

We begin with some preparations needed for all of the proofs of the different claims of the proposition, this is part 0, followed by three further parts explained at the end of part 0.

Part 0 (Preparation)

For the proof we represent the process differently, namely we rewrite the problem in measure-valued form. We will need the following functional of the process Π_t^N, denoted $\widehat{\Pi}_t^N$ respectively $\widehat{\Pi}_t^{N,k}, \widehat{\Pi}_t^{N,(k,\ell)}$ if we indicate the initial state in the notation (here $s_i = s_i^N$):

$$(3.321) \qquad \widehat{\Pi}_t^N = \sum_{i=1}^{K_t^N} \zeta_{s_i}^N(t) 1(\zeta_{s_i}^N(t) > 1),$$

which counts at time t only those dual particles which are not the only ones at their site, i.e. they are the ones which can generate new occupied sites. Here $\zeta_s^N(t)$ is the single site birth and death process starting at time s, the time where the site was first occupied, with mean-field immigration at time t given by $c\frac{\Pi_t^{N,1}}{N} - o(1)$ as $N \to \infty$ from the other sites.

Note that we have now made the N-dependence explicit. Also note that by definition of U^N, u^N and $\widehat{\Pi}^N$ we have

$$(3.322) \qquad \widehat{\Pi}_t^{N,1} = u^N(t) \int_0^\infty \sum_{j=2}^\infty j U^N(t, ds, j).$$

To continue we introduce a new random object from which we can read off (u^N, U^N) by taking suitable functions. Namely we consider $\Psi^N(t, ds, dx)$ the $\mathcal{M}_{\text{fin}}([0,\infty) \times \mathbb{N}_0)$- valued random variable (finite measures on $[0,\infty) \times \mathbb{N}_0$) defined by

$$(3.323) \qquad \Psi^N(t, ds, dx) = \sum_{s_i \le t} \delta_{(t - s_i, \zeta_{s_i}^N(t))},$$

where $\{s_i\}$ (recall $s_i = s_i^N$) denote the times of birth of new sites and i runs from 1 to K_t^N (this set of times depends on N). Note that here *empty sites are not counted.* We know from the CMJ-theory that $\Psi^N(t, \cdot, \cdot)$ is bounded by $W^* e^{\alpha t}$. Since we are interested in times $\alpha^{-1}\log N + r$ we introduce the scaled object:

$$(3.324) \qquad \widehat{\Psi}_t^N = \frac{\Psi_t^N(\cdot,\cdot)}{N}.$$

Note that for fixed N this is an element of $\mathcal{M}_{\text{fin}}([0,\infty)\times\mathbb{N}_0)$.

Next note that indeed we obtain our pair (u^N, U^N) as a functional of Ψ^N. Observe:

$$u^N(t)=\Psi_t^N([0,\infty)\times\mathbb{N}),\quad U^N(t,\cdot,\cdot)=\frac{\Psi_t^N(\cdot,\cdot)}{u^N(t)}=\frac{\widehat{\Psi}_t^N(\cdot,\cdot)}{u^N(t)/N} \tag{3.325}$$

and the stochastic integral below gives:

$$\Pi_t^N=\int_0^t \zeta_u^N(t)dK_u^N=\int_0^t\int_0^\infty x\Psi_t^N(ds,dx). \tag{3.326}$$

Therefore from (3.325) we obtain the convergence of $(N^{-1}u^N,U^N)$ in the time scale $\alpha^{-1}\log N+t$ with $t\in\mathbb{R}$ if we show the convergence of $\widehat{\Psi}^N$. This will also allow us later on to argue that the mass and time scaled process

$$\bar{\Pi}^N(t)=N^{-1}\Pi^N_{\frac{\log N}{\alpha}+t} \tag{3.327}$$

converges as $N\to\infty$ to a limit.

The proof is broken into three main parts. The first two parts concern the pair (u,U) the third one the quantity $N^{-1}\Pi^{N,k,\ell}_{T_N+t}$. The parts can be outlined as follows (where $(a),\cdots$ refers to the parts of the proposition):

1. Assuming the marginals of $(N^{-1}\widetilde{u}^N,\widetilde{U}^N)$ converge in distribution at $t=t_0$, prove (a) that the limiting evolution during $t\geq t_0$ follows the deterministic nonlinear dynamics specified in the system of coupled equations (3.248), (3.249) and that pathwise convergence holds.
2. Then prove (b) that the marginal distributions converge for $t_0=t_0(N)=\frac{1}{\alpha}\log N+s, s\in\mathbb{R}$ and show (c), (d) on the corresponding entrance behaviour.
3. Show the convergence of the process $\{N^{-1}\Pi^{N,k,\ell}_{T_N+t},t\in\mathbb{R}\}$ and of its time integrals to complete the proof of (e).

Part 1 (Convergence of dynamics of $\widehat{\Psi}^N$)

Let $P^N\in\mathcal{P}(D([t_0,\infty),\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})))$ denote the law of the Markov process

$$\left\{\widehat{\Psi}^N_{T_N+t}\right\}_{t\geq t_0}\text{ (recall (3.324)) with }T_N=\tfrac{\log N}{\alpha}. \tag{3.328}$$

In this part we assume that the marginal distributions $P^N_{t_0}$ converge as $N\to\infty$. Our next goal is then to show that the processes converge in law on path space, that is, $P^N\Rightarrow P$ as $N\to\infty$ where P is the law of a deterministic dynamics.

Recall that we have defined the candidate for the limiting object by a nonlinear equation, which we can view as forward equation for a nonlinear Markov process on $\mathbb{R}_+\times\mathbb{N}$. This means we want to show that if we consider the marginal laws, that is, elements P_t^N of $\mathcal{P}(\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N}))$ they converge to an element $\delta_{\widehat{\Psi}_t}$ with $\widehat{\Psi}_t\in\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})$. We will now take the viewpoint of backward equations and their generators, that is equations for expectations of measures rather than actual weights.

In order to do so in this present Part 1, we proceed as follows.

- we obtain moment bounds on $\widehat{\Psi}^N_{T_N+t}$ uniform for $N\in\mathbb{N}$ and $t_0\leq t\leq t_0+T$,
- we formulate the martingale problem for $\widehat{\Psi}^N$ with generator $\mathbf{G}_N$ given by a linear operator on the space $C_b(\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N}_0)))$ with common domain $\mathcal{D}$ given by a certain dense subset of $C_b(\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N}_0))$,

- we verify the tightness of the laws P^N,
- we prove the weak convergence of P^N to P by showing that $\mathbf{G}_N$ converges pointwise to $\mathbf{G}$ on these functions in the domain $\mathcal{D}$ where $\mathbf{G}$ is the operator defined in (3.259),
- verify that limit points of $(P^N)_{N\in\mathbb{N}}$ are characterized as solutions to the $\mathbf{G}$ martingale problem,
- conclude the convergence since we have proved that the $\mathbf{G}$ martingale problem has a unique solution given by the nonlinear system (3.248), (3.249),
- the convergence for fixed t_0 is obtained later in Part 2.

Abbreviate for bounded measureable test functions f on $[0,\infty)\times\mathbb{N}_0$

$$\widehat{\Psi}(f)=\int\limits_0^\infty\int\limits_\infty^\infty f(r,x)\widehat{\Psi}(dr,dx) \tag{3.329}$$

and in particular $\widehat{\Psi}(1)=\widehat{\Psi}([0,\infty)\times(\mathbb{N}_0))$.

The process $\widehat{\Psi}^N$ is a Markov process with state space $\mathcal{M}_{\text{fin}}([0,\infty)\times\mathbb{N})$. We calculate the action of the generator $\mathbf{G}_N$ acting on the domain $\mathcal{D}$ of functions $F_{g,f}$ of the form (3.255) as follows. Let $\widehat{\Psi}\in\mathcal{M}_{\text{fin}}([0,\infty)\times\mathbb{N})$. Then:

$$\begin{aligned}
&(\mathbf{G}_N F_{g,f})(\widehat{\Psi})=\\
&c(1-\widehat{\Psi}(1))\left(\int_0^\infty\int_0^\infty N\left[g(\widehat{\Psi}(f)+\tfrac{f(0,1)+f(r,x-1)-f(r,x)}{N})-g(\widehat{\Psi}(f))\right]x1_{x\neq 1}\widehat{\Psi}(dr,dx)\right)\\
&+\int\limits_0^\infty\int\limits_1^\infty g'\left(\widehat{\Psi}(f)\right)\tfrac{\partial f(r,x)}{\partial r}\widehat{\Psi}(dr,dx)\\
&+\int\limits_0^\infty\int\limits_1^\infty sxN\left[g(\widehat{\Psi}(f)+\tfrac{f(r,x+1)-f(r,x)}{N})-g(\widehat{\Psi}(f))\right]\widehat{\Psi}(dr,dx)\\
&+\int\limits_0^\infty\int\limits_1^\infty \tfrac{d}{2}x(x-1)N\left[g(\widehat{\Psi}(f)+\tfrac{f(r,x-1)-f(r,x)}{N})-g(\widehat{\Psi}(f))\right]\widehat{\Psi}(dr,dx)\\
&+c\widehat{\Psi}(1)\int\limits_0^\infty\int\limits_1^\infty\int\limits_0^\infty\int\limits_1^\infty N\left[g(\widehat{\Psi}(f)+\tfrac{f(\tilde r,\tilde x+1)-f(\tilde r,\tilde x)}{N}+\tfrac{1_{x\neq 1}f(r,x-1)-f(r,x)}{N})-g(\widehat{\Psi}(f))\right]\\
&\qquad\qquad x\widehat{\Psi}(dr,dx)\tfrac{\widehat{\Psi}(d\tilde r,d\tilde x)}{\widehat{\Psi}(1)}.
\end{aligned} \tag{3.330}$$

Note that in order for the right side to be well-defined we require $\int x^2\widehat{\Psi}(\mathbb{R},dx)<\infty$. This condition is automatically satisfied if $\widehat{\Psi}\in\mathcal{M}_{\text{fin}}([0,\infty)\times\mathbb{N}_0)\cap\mathbb{B}$. This means that $\mathbf{G}_N F_{g,f}$ is a welldefined function on $\mathbb{B}$, where it is even continuous. Note however $C_b(\mathbb{B},\mathbb{R})$ is *not* mapped in $C_b(\mathbb{B},\mathbb{R})$. This means we now have to investigate next that under our evolution we stay with $\widehat{\Psi}$ being $\mathbb{B}$. Therefore we turn next to bounds on the empirical moments of the number of particles at the occupied sites.

We now establish some apriori bounds which will guarantee that the empirical moments remain stochastically bounded over the considered time interval. By definition we have

$$\int_0^\infty x^m\widehat{\Psi}^N_{T_N+t}(\mathbb{R},dx)=\frac{1}{N}\sum_{i=1}^{u^N(T_N+t)}\sum_{k=1}^\infty k^m 1_k(\zeta^N_{s_i}(T_N+t)). \tag{3.331}$$

Then we have with s_i denoting the time of birth of the i-th colonized site (recall $s_i=s_i^N$) and letting S be the birth time of a randomly picked site:

$$E\left[\int_0^\infty x^m \widehat{\Psi}^N_{T_N+t}(\mathbb{R},dx)\right] = \frac{1}{N}E\left[\sum_{i=1}^{u^N(T_N+t)}\sum_{k=1}^{\infty} k^m 1[(\zeta^N_{s_i}(T_N+t)=k)]\right] \tag{3.332}$$
$$= E\left[\frac{1}{N}\sum_{i=1}^{u^N(T_N+t)}(\zeta^N_{s_i})^m(T_N+t)\right] \leq E\left[(\zeta^N_S(T_N+t))^m\right]$$

and by Cauchy-Schwarz applied to the m-th process of $\zeta^N_{s_i}$ we get

$$E\left[\left(\int_0^\infty x^m \widehat{\Psi}_{T_N+t}(\mathbb{R},dx)\right)^2\right] \tag{3.333}$$
$$\leq \left(E\Big[\frac{1}{N}\sum_{i=1}^{u^N(T_N+t)}(\zeta^N_{s_i}(T_N+t))^m\Big]\right)^2 + E\Big[\frac{1}{N}\sum_{i=1}^{u^N(T_N+t)}(\zeta^N_{s_i}(T_N+t))^{2m}\Big].$$

We can obtain an upper bound for $\int_0^\infty x^m \widehat{\Psi}^N_{T_N+t}(\mathbb{R},dx)$ by comparing with the collision free case (recall construction in Step 2 of Subsubsection 3.2.6 that is, the total number of particles is bounded by the collision free case) to obtain

$$\lim_{L\to\infty} P\left[\sup_{t_0\leq t\leq t_0+T}\int_0^\infty x\widehat{\Psi}^N_{T_N+t}(\mathbb{R},dx) > L\right] \tag{3.334}$$
$$\leq \lim_{L\to\infty} P\left[\frac{1}{N}\sum_{i=1}^{u_N(T_N+t)}\sup_{t_0\leq t\leq t_0+T}\zeta^N_{s_i}(T_N+t) > L\right]$$
$$\leq \lim_{L\to\infty}\frac{E\left[\sup_{t_0\leq t\leq t_0+T}(\zeta_s(t))^2\right]}{L^2} = 0,$$

using the independence of the $\{\zeta_s\}$ in the collision-free case and the martingale problem together with a standard martingale inequality for the birth and death process $\zeta_s(\cdot)$.

We have seen that the first empirical moment bound (3.334) holds by comparison with the collision-free regime. We now have to establish analogous results for higher empirical moments and incorporating collisions which involves additional immigration from the other colonies at each occupied site. Let $K > 0$. Let

$$\zeta^{N,K}_{s_i}(t) \tag{3.335}$$

denote the birth and death process with immigration at rate K which starts at time s_i. This process is ergodic and has a unique equilibrium. Let $m_n(K)$ denotes the corresponding n-th equilibrium moment.

Remark 27 *We used above the fact that the birth and death process with linear birth and quadratic death rates has moments of all orders at positive times. This can be verified by comparing with a subcritical branching process with immigration. Moreover the corresponding Markov process on* $\mathbb{N}$ *has an entrance law starting at* $+\infty$*. This property is wellknown for the Kingman coalescent but extends to our birth and death process provided* $d > 0$ *and is easily extended to include an additional immigration term.*

Then up to the time

$$\tau_K := \inf\{t : \int_0^\infty x\widehat{\Psi}_{T_N+t}(\mathbb{R}, dx) \geq K\} \tag{3.336}$$

we can verify with a stochastic montonicity argument that we get a bound uniformly in N:

$$E[(\zeta_{s_i}^{N,K}(\cdot))^n] \leq m_n(K) < \infty, \quad \text{for } n, N \in \mathbb{N}. \tag{3.337}$$

Now recall that (writing $F_{x,f}$ if $g(x) = x$)

$$M_t^N := \widehat{\Psi}_{T_N+t}^N(f) - \widehat{\Psi}_{T_N+t_0}^N(f) - \int_{T_N+t_0}^{T_N+t} G_N F_{x,f}(\widehat{\Psi}_{T_N+s}^N(\mathbb{R}, dx))ds \tag{3.338}$$

is a martingale. Using the above moment inequalities, (see (3.333)), we can show that this is an L^2 *martingale* and we can then use a martingale inequality to show that for f with finite support

$$\begin{aligned} &\lim_{L\to\infty} \sup_N P(\sup_{t_0 \leq t \leq (t_0+T)\wedge\tau_K} \widehat{\Psi}_{T_N+t}^N(f) > L) \\ &\leq \lim_{L\to\infty} \sup_N \frac{4}{L^2} \left\{ E\left[\left(\widehat{\Psi}_{T_N+t_0}^N(|f|) + \int_{T_N+t_0}^{T_N+T} |G_N F_{x,f}(\widehat{\Psi}_{T_N+s}^N(\mathbb{R}, dx))| ds\right)^2\right]\right\} \\ &+ \lim_{L\to\infty} \sup_N \frac{4}{L^2} E[(M_{(t_0+T)\wedge\tau_K}^N)^2] \\ &= 0. \end{aligned} \tag{3.339}$$

For every $n \geq 2$ we can now choose a sequence $(f_m)_{m\in\mathbb{N}}$ with finite support in $C(\mathbb{R}, \mathbb{R})$ and let $f_m(x) \uparrow x^n$ verify that for given K

$$\lim_{L\to\infty} \sup_N P\left[\sup_{t_0 \leq t \leq (t_0+T)\wedge\tau_K} \int_0^\infty x^n \widehat{\Psi}_{T_N+t}^N(\mathbb{R}, dx) > L\right] = 0. \tag{3.340}$$

We next note that equation (3.334) implies that

$$\tau_K^N \uparrow \infty \text{ as } K \to \infty \tag{3.341}$$

uniformly in N. This together with (3.340) implies that the evolutions $\{\widehat{\Psi}_t^N, \quad t \in [T_N + t_0, T_N + t_0 + T]\}, \quad N \in \mathbb{N}$ and their weak limit points are concentrated on *path with values in* $\mathbb{B}$.

We can conclude now with noting that $\mathbf{G}_N(F_{g,f})$ is a finite and even continuous function on $\mathbb{B}$, even though $\mathbf{G}_N$ does not map functions in its domain $\subseteq C_b(\mathbb{B}, \mathbb{R})$ into $C_b(\mathbb{B}, \mathbb{R})$ but only into the set $C(\mathbb{B}, \mathbb{R})$ of *continuous functions* on $\mathbb{B}$. Therefore we will have to show first that for the starting points of the evolution we work with, namely the processes $\{\widehat{\Psi}^N(t), t \in [t_0, T]\}$ and the limit points of their laws are concentrated on states satisfying this extra condition to be in $\mathbb{B}$. This is now immediate from the a priori bound (3.340).

We now verify first tightness in path space and then in a further step convergence of the distribution.

Tightness We verify the *tightness* in path space that is the tightness of the laws

$$\{P^N = \mathcal{L}[(\widehat{\Psi}_t^N)_{t\geq t_0}]; N \in \mathbb{N}\} \text{ as } N \to \infty, \tag{3.342}$$

assuming convergence of the marginals at time t_0. Here we deal with a sequence of probability measure-valued (on a Polish space) jump processes. With the help of Jakubowski's criterion, (see Theorem 3.6.4 in [D]) we obtain that it suffices to consider the tightness of laws of real

valued semimartingales, namely of $\{\mathcal{L}[(F_{g,f}(\widehat{\Psi}^N_t))_{t\geq 0}]$, admissible f and $g\}$. In fact $g(x)=x$ and $g(x)=x^2$ (recall Remark 23) alone suffice according to that theorem and for these two functions g and all f with $0\leq f\leq 1$ we shall now verify the criterion.

Recall first the apriori bound (3.340). The tightness of the laws of $F_{g,f}\circ\widehat{\Psi}^N$ is is obtained using the Joffe-Métivier criterion ([JM]) in the form of Corollary 3.6.7. in [D], applied to $F_{g,f}$ for $g(x)=x$ and $g(x)=x^2$. The bound (3.340) verifies the conditions therein. This means that now $\{\mathcal{L}[(\Psi^N_{T_N+t})_{t\geq t_0}], N\in\mathbb{N}\}$ has limit points in the set of laws on Skorohod space.

Let $P^N_{\widehat{\Psi}^N_0}$ denote the law of $\{\widehat{\Psi}^N_0\}_{t\geq t_0}$ on the Skorohod space $D([0,\infty),\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B})$ with initial values $\widehat{\Psi}^N_{t_0}\Rightarrow\widehat{\Psi}_{t_0}\in\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B}$. By tightness we can choose a convergent subsequence. Since the largest jumps decrease to 0 the limit measure is automatically concentrated on $C([0,\infty),\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B})$ (see [EK2], Chap.3, Theorem 10.2). Convergence in law on the Skorohod space follows if any limit point satisfies a martingale problem with a unique solution.

Convergence Next we turn to the convergence in f.d.d. by establishing *generator convergence.* For this purpose we consider $\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B}$ as the state space for the processes $\widehat{\Psi}_N(\cdot)$ and the *function space* $C_b(\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B},\mathbb{R})$ as the basic space for our semigroup action. Recall the algebra of functions $D_0(\mathbf{G})$ given in (3.259). The operator $\mathbf{G}_N$ is a *linear map* from $D_0(\mathbf{G})$ into $C(\mathcal{M}_{\text{fin}}(\mathbb{R}_+\times\mathbb{N})\cap\mathbb{B},\mathbb{R})$.

We consider the convergence for two choices for the "initial state" at some time $t_0=t_0(N)$, namely we assume that $\widehat{\Psi}^N_{t_0}$ converges as $N\to\infty$, but either

$$\widehat{\Psi}^N_{t_0}(1)\underset{N\to\infty}{\longrightarrow}0\quad\text{ or }\quad\widehat{\Psi}^N_{t_0}(1)\underset{N\to\infty}{\longrightarrow}x>0. \tag{3.343}$$

In the first case we would be back in the linear regime (no collisions) we discussed in Subsubsection 3.2.7 and which we have identified in terms of the CMJ process. Next we consider the limit corresponding to initial points at some time t_0 such that $\widehat{\Psi}^N(t_0)$ does not converge to the zero measure.

Let $f\geq 0$, $\|f\|_\infty\leq 1$, $\widehat{\Psi}(1)\leq 1$, g with $g|_{([0,1],\mathbb{R})}\in C^2_b([0,1],\mathbb{R})$ and $F_{g,f}$ as in (3.255). Recall the definition of the operator $\mathbf{G}$ on $D_0(\mathbf{G})$ given in (3.259). Then there exists a constant (const) depending only on c,d,s such that:

$$|\mathbf{G}_N(F_{g,f})(\widehat{\Psi})-\mathbf{G}(F_{g,f})(\widehat{\Psi})|\leq\frac{\text{const}}{N}\|g''1_{[0,1]}\|_\infty\int_0^\infty\int_0^\infty x^2\widehat{\Psi}(dr,dx)\leq\frac{\text{const}}{N}\|\widehat{\Psi}\|_{\mathbb{B}}. \tag{3.344}$$

We get that for all $F_{g,f}$ of the form in (3.255) with the above extra restrictions that

$$\mathbf{G}_N(F_{g,f})(\widehat{\Psi})\underset{N\to\infty}{\longrightarrow}\mathbf{G}(F_{g,f})(\widehat{\Psi}), \tag{3.345}$$

provided that the argument $\widehat{\Psi}$ satisfies

$$\int_0^\infty\int_0^\infty x^2\widehat{\Psi}(dr,dx)<\infty, \tag{3.346}$$

which is the case for $\widehat{\Psi}\in\mathbb{B}$. More precisely, considering the functions $\mathbf{G}_N(F_g),\mathbf{G}(F_g)$ on any finite ball in $\mathbb{B}$, we have

$$\mathbf{G}_N(F_{g,f})\longrightarrow\mathbf{G}(F_{g,f}),\quad N\to\infty \tag{3.347}$$

pointwise uniformly on such balls.

From (3.347) and the convergence of the initial laws we want to conclude that weak limit points of $\mathcal{L}[(\widehat{\Psi}^N_t)_{t\geq t_0}]$ conditioned on a particular initial state at time t_0 are δ-measures on the space of $\mathcal{M}_{\text{fin}}(\mathbb{R}^+\times\mathbb{N})$-valued path which must satisfy the evolution equation

$$\frac{d}{dt}(\widehat{\Psi}_t)=G^*\widehat{\Psi}_t\quad,\quad t\geq t_0. \tag{3.348}$$

For this we have to show

1. the variance process of $< \widehat{\Psi}^N, f >$ converges to zero, and
2. $\widehat{\Psi}^N$ remains in large enough balls with high probability such that (3.347) can be applied,
3. making the choice $g(x) = x$ in (3.255), to verify that the path solves equation (3.348),
4. to prove convergence it then suffices to show that the **G**-martingale problem is wellposed.

For the point (1) consider $g(x) = x^2$ and $g(x) = x$ and show that

$$\lim_{N\to\infty} [\mathbf{G}_N(F_{x^2,f}) - 2F_{x,f}\mathbf{G}_N F_{x,f}] = 0. \tag{3.349}$$

Note that replacing G_N by G the relation says that G is a *first order* operator and this is read of from the form of G directly. This can be verified also directly using (3.259) and (3.344). This implies that the variance of a solution to the limiting martingale problem is 0. Since the higher moments of $\langle \widehat{\Psi}^N, f\rangle$ are bounded this implies (1). Point (2) is given by the apriori estimate (3.340).

For point (3) we take $g(x) = x$ and conclude that the limit point is given by the unique trajectory in $\mathbb{B}$ which satisfies

$$\int_0^\infty \int_0^\infty f(r,x)\widehat{\Psi}_t(dr,dx) - \int_0^t \int_0^\infty \int_0^\infty G^{\widehat{\Psi}_s} f(r,x)\widehat{\Psi}_s(dr,dx)ds = 0, \tag{3.350}$$

that is, the nonlinear system (3.254) (recall Lemma 3.6). In other words, any limit point is a solution to the **G** martingale problem. For point (4), we have established the uniqueness and properties of the solution to the **G** martingale problem in Lemma 3.6 and Corollary 3.7.

To complete the proof of convergence, given $\widehat{\Psi}_{t_0}$ and T, find $K_0 > 0$ such that the solution of the nonlinear system stays in the interior of the ball of radius K_0 during the interval $[t_0, t_0 + T]$ and let $K \geq 2K_0$, say. Then it follows from the above that we have weak convergence up to time $(t_0 + T) \wedge \tau_K$ for every $K > 0$ and therefore for $\varepsilon > 0$

$$\lim_{N\to\infty} P[\sup_{t_0 \le t \le t_0+T} \|\widehat{\Psi}_t^N - \widehat{\Psi}_t\| > \varepsilon] = 0. \tag{3.351}$$

This completes the proof of part (a) of the proposition.

Part 2 Convergence of t_0-marginals and properties of (u^N, U^N)

Our goal is to prove part (b) and (c) of the proposition under the assumptions of convergence at a fixed time $\alpha^{-1}\log N + t_0$.

Proof of (1)

For that purpose we prove first the convergence of the marginals at times $\alpha^{-1}\log N + t_0$, starting with "$t_0 = -\infty$", (3.311), and then later consider $t_0 \in \mathbb{R}$ to finally come to the convergence of the path.

Proof of (3.311) Here we have a time scale where as $N \to \infty$ the collisions become negligible since the observation time diverges sublogarithmically as function of N. Using the multicolour particle system (see Subsubsection 3.2.6) we can build all variables on one probability space. From this construction it follows that collisions become negligible and we have the claimed convergence from the one for the collision-free system proved in Subsubsection 3.2.4, see (3.143), and Subsubsection 3.2.5 see (3.166) via the CMJ-theory. This proves part (b) of the proposition.

Proof of (c). We now have to prove the convergence result for the marginal of $(\widehat{\Psi}_t^N)_{t\in\mathbb{R}}$ at times $t = \alpha^{-1}\log N + t_0$, where we can choose $t_0 \in \mathbb{R}$ as small as we want since we know that the dynamic converges to a continuous limit dynamic. This means we have to show (3.312), (3.313)

and (3.314). Finally we have to relate the limit of $(N^{-1}\widetilde{u}^N, \widetilde{U}^N)$ as $N \to \infty$ to the system of nonlinear equations to then prove (3.316) and (3.317).

Proof of (3.312) and (3.313) We recall that we proved in the previous part the convergence of $N^{-1}\widetilde{u}^N$ pathwise if we consider path for $t \geq t_0$ and if we have convergence in law at $t = t_0$. Therefore it suffices now to show that $N^{-1}\widetilde{u}^N(t_0)$ converges (in law) as $N \to \infty$.

To do this we first note that the time-scaled and normalized (i.e. scaled by N^{-1}) number of sites occupied at time $\alpha^{-1}\log N + t_0$, denoted $N^{-1}\widetilde{u}^N$, does not go to zero as $N \to \infty$ nor does it become unbounded. This was proved in Proposition 1.7. This proves tightness of the one-dimensional marginal distributions at times $\alpha^{-1}\log N + t$ and that limit points are non-degenerate.

The tightness gives us the existence of limiting laws for $(N^{-1}\widetilde{u}^N, \widetilde{U}^N)$ along subsequences for time $\alpha^{-1}\log N + s$ for fixed s. Since $N^{-1}\widetilde{u}^N \leq We^{\alpha t}$ any limit point must satisfy

$$\limsup_{t \to -\infty} e^{-\alpha t} u(t) < \infty \text{ and } \|U(t)\| \text{ bounded.} \tag{3.352}$$

Therefore by Lemma 3.6(d) any limit point corresponds to a solution to the nonlinear system given by (3.281) and (3.282) and any two such solutions are (random) *time shifts* of each other. Hence we have to show that each pair of limit prints belongs to the same time shift from the standard solution of the system with growth constant 1.

Recall the definitions of $\tau^N(\varepsilon)$ (which is the first time where $u_s^{N,k,\ell}$ reaches $\lfloor \varepsilon N \rfloor$ for all $\varepsilon \in (0,1)$), $\widetilde{\tau}^N(\varepsilon)$ from (3.183), (3.184). Now assume that there are two limit points and let the corresponding limiting passage times be denoted $\tau^1(\varepsilon)$ and $\tau^2(\varepsilon)$ such that $\tau^{N_{n_i},i}(\varepsilon) \to \tau^i(\varepsilon)$, $i = 1,2$ in law. But by (3.221), $P[|\tau^{N_{n_2},2}(\varepsilon) - \tau^{N_{n_1},1}(\varepsilon)| > 2C(\varepsilon)] \to 0$. But since $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, the time shift must be arbitrarily small, which means equal to zero and therefore the two limit points coincide. This immediately proves that $\tau^N(\varepsilon)$ converges for any $\varepsilon > 0$.

We now elaborate on this argument. To show convergence of the marginal distributions it suffices to prove the convergence in law of the *normalized* first passage times $\tau^N_{\mathrm{norm}}(\varepsilon) := \tau^N(\varepsilon) - (\alpha^{-1}\log N)$. We give two arguments which verify that convergence of the $\tau^N_{\mathrm{norm}}(\varepsilon)$ is sufficient.

(i) Given the value $u(s)$ we know exactly the first passage time for all $\varepsilon > u(s)$ (due to the *unique* and also *deterministic* evolution from any initial point). Indeed using the fact that the path converges (for convergent sequences at a fixed time), the collection is monotone non-decreasing and cadlag, we know that the whole collection (in ε) of first passage time converges in path space. The uniqueness of the limit points follows since two different limiting distributions at times $\alpha^{-1}\log N + s$ for fixed s would not be compatible with the uniqueness in distribution of the limiting first passage times, since the limiting dynamics $(u(t))_{t \in \mathbb{R}}$ is monotone for sufficiently small (i.e. negative t) so that it is uniquely determined by the first passage times in that range and hence everywhere.

(ii) Alternatively we can use the fact that there is according to Lemma 3.6 up to a time shift a unique entrance law from 0 at $t \to -\infty$ which is compatible with the growth conditions of a limit of that quantity as $N \to \infty$. Therefore by one first passage time this random time shift is determined and hence it suffices to prove convergence in law of one first passage time concluding the second argument.

Having proved the reduction of the convergence statement to one on τ^N_{norm}, we have to show next that for one $\varepsilon \in (0,1)$ and therefore by the above for every such ε:

$$\mathcal{L}[\tau^N(\varepsilon) - \frac{\log N}{\alpha}] \underset{N \to \infty}{\Longrightarrow} \mathcal{L}[\tau(\varepsilon)], \tag{3.353}$$

with

$$\tau(\varepsilon) = \inf(t \in \mathbb{R} | u(t) = \varepsilon), \tag{3.354}$$

where u is the (unique) entrance law with $e^{\alpha|t|}u(t) \underset{t \to -\infty}{\longrightarrow} W$ (recall Lemma 3.6).

We have two tasks, to identify the limit on the r.h.s. of (3.353) as the quantity defined in (3.354) this however we showed in part 1 where we established in particular the convergence of the dynamic for the frequency of occupied sites, and secondly we have to show the convergence in (3.353). So only the latter remains.

We use our results on the corresponding times $\widetilde{\tau}^N(\varepsilon)$, assuming no collisions occur, namely we see that it follows from Proposition 3.5 that $\widetilde{\tau}^N(\varepsilon)$ converges to a limit $\widetilde{\tau}(\varepsilon)$ defined in terms of the *collision-free* process. Hence it remains to bridge from $\tau^N(\varepsilon)$ to $\widetilde{\tau}^N(\varepsilon)$.

To relate $\widetilde{\tau}^N(\varepsilon)$ and $\tau^N(\varepsilon)$ we will use that we can use according to the above arguments arbitrarily small $\varepsilon > 0$ and we can work with the multicolour particle system which we introduced in the proof of Proposition 1.7 below (3.176).

Consider $\varepsilon \downarrow 0$. We have the fact from the multicolour particles system (recall (3.195) and (3.197)) that for sufficiently small ε asymptotically as $N \to \infty$ in probability

$$\widetilde{\tau}(\varepsilon) \leq \tau^N(\varepsilon) \leq \tilde{\tau}(\widetilde{\varepsilon}), \tag{3.355}$$

where $\widetilde{\varepsilon}$ solves

$$\widetilde{\varepsilon} - \text{ Const } \widetilde{\varepsilon}^2 = \varepsilon, \quad \widetilde{\varepsilon} > 0. \tag{3.356}$$

Note that the Const is less than 1, so that there is a unique $\widetilde{\varepsilon} > \varepsilon$ such that

$$\widetilde{\varepsilon} - \varepsilon = O(\widetilde{\varepsilon}^2). \tag{3.357}$$

In (3.355) the right inequality holds for ε small enough and N *sufficiently large*, while the l.h.s. holds even for every N.

Denote by $\tau^N_{\text{norm}}(\varepsilon)$ the normalised $\tau^N(\varepsilon)$, i.e. $\tau^N(\varepsilon) - \alpha^{-1}\log N$. We know that $\widetilde{\tau}^N(\varepsilon) \longrightarrow \widetilde{\tau}(\varepsilon)$ as $N \to \infty$ if we use the coupled collection of processes we constructed in the proof of equation (3.312). Hence for ε_0 sufficiently small realizations of two different limit points $\tau^{\infty,1}_{\text{norm}}(\varepsilon_0), \tau^{\infty,2}_{\text{norm}}(\varepsilon_0)$ satisfy on a suitable probability space:

$$\widetilde{\tau}(\varepsilon) \leq \tau^{\infty,1}_{\text{norm}}(\varepsilon), \tau^{\infty,2}_{\text{norm}}(\varepsilon) \leq \widetilde{\tau}(\widetilde{\varepsilon}). \tag{3.358}$$

Therefore $|\tau^{\infty,1}_{\text{norm}}(\varepsilon) - \tau^{\infty,2}_{\text{norm}}(\varepsilon)| \leq |\widetilde{\tau}(\varepsilon) - \widetilde{\tau}(\widetilde{\varepsilon})|$. We furthermore know that $\widetilde{\tau}(\varepsilon)$ is given by $(\log \varepsilon - \log W)/\alpha$ so that the l.h.s. and the r.h.s. of (3.358) differ by a *deterministic* quantity, which equals $const \cdot \widetilde{\varepsilon}_0$ as $\widetilde{\varepsilon}_0 \downarrow 0$. As $N \to \infty$ we have a limiting evolution between $\tau^N(\widetilde{\varepsilon}_0)$ and $\tau^N(\widetilde{\varepsilon})$ which is deterministic. This evolution is a *random shift* of some deterministic curve. Then it is not possible to have two limit points for the law $\tau^N(\widetilde{\varepsilon})$, since their difference are shifted versions of a given curve would translate into the same difference but now bounded above by arbitrarily small $\widetilde{\varepsilon}_0$ and therefore equal to 0. Hence all limit points are equal and $\tau^N_{\text{norm}}(\varepsilon) = (\tau^N(\varepsilon) - \alpha^{-1}\log N)$ must converge as $N \to \infty$ as claimed and we are done. This concludes the argument for the convergence of the marginals at time $\alpha^{-1}\log N + t_0$.

Proof of (3.314) We first need some preparation, involving the collision-free regime. Turn to the dual particle process and recall that at time $\alpha^{-1}\log\log N$ asymptotically as $N \to \infty$ there are $W \log N$ occupied sites and that at time $\alpha^{-1}\log\log N$ the empirical age and size distribution is given by the CMJ stable age and size distribution $\mathcal{U}(\infty, \cdot, \cdot)$, since we are in the non-collision regime due to $\log\log N << \alpha^{-1}\log N$. Therefore at time $\tau_{\log N}$ the age distribution is (asymptotically as $N \to \infty$) given by the stable age distribution. Moreover for each realization of W there is a time shift of the limiting dynamics which is a function of W. Since the evolution from this point $\tau_{\log N}$ on is asymptotically deterministic we conclude that the age and size distribution U^N at time $\alpha^{-1}\log N - t_N + t$, with $t_N \uparrow \infty$ but $t_N << \alpha^{-1}\log N$, converges to a time shift of the one determined by equation (3.248) and (3.249) with initial condition given by $u = 0$ and U given by the stable age distribution. We now have to bridge from times $\alpha^{-1}\log N - t_N + t$ to $\alpha^{-1}\log N + t$, where collisions play a role. This runs as follows.

For a given approximation parameter $\varepsilon > 0$ we choose a time horizon $\alpha^{-1}\log N + T(\varepsilon)$ such that $(K_t^N, \zeta_t^N)_{t\geq 0}$ is, as $N \to \infty, \varepsilon$-approximated by the collision-free process $(K_t, \zeta_t)_{t\geq 0}$ up to time $\alpha^{-1}\log N + T(\varepsilon)$ if we choose ε sufficiently small and this implies the approximation for the functionals U^N by $\mathcal{U}$ as $N \to \infty$ for ε sufficiently small. More precisely we require that we can find a coupling between U^N and $\mathcal{U}$ such that ($\|\cdot\|$ denotes variational distance)

$$\limsup_{N\to\infty} P\left(\sup_{t\leq \frac{\log N}{\alpha}+T(\varepsilon)} \|U^N(t,\cdot,\cdot) - \mathcal{U}(t,\cdot,\cdot)\| > \varepsilon\right) \to 0 \text{ as } \varepsilon \to 0. \tag{3.359}$$

In other words we have to show that by the collisions up to time $\alpha^{-1}\log N + T(\varepsilon)$ at most difference ε develops between U^N and $\mathcal{U}$ for very large N.

To show this note that we can bound the total effect of collisions occurring up to time $\alpha^{-1}\log N + T(\varepsilon)$. Namely we return to the multicolour particle system defined below (3.176), which shows that we can relate the difference of the system without and with collision through the black and red particles. In order to show that $\|U^N(t,\cdot,\cdot) - \mathcal{U}(t,\cdot,\cdot)\| \leq \varepsilon$ it suffices to show that the proportion or red and black particles is $o(\varepsilon)$ in probability as $N \to \infty$. But this was verified for the number of black sites with $T(\varepsilon) = \widetilde{\tau}(\varepsilon)$ in (3.195), which immediately by using that ζ is stochastically bounded gives the claim for the number of particles. Since the number of descendants of red particles grows at a slower rate as the black and new red founding particles occur at the same rate as black ones then the claim follows. This completes the proof of Part (c).

Proof of (d): (3.316) and (3.317)

Again we first turn to the collision-free process. We approximate (recall $\mathcal{U}$ refers to the collision-free CMJ-process) $\mathcal{U}(t,\cdot,\cdot)$ by $\mathcal{U}(\infty,\cdot,\cdot)$. By [N] (Theorem 6.3) we know that $\mathcal{U}(r,\cdot,\cdot)$ converges a.s. as $r \to +\infty$ to the stable age and size distribution $\mathcal{U}(\infty,\cdot,\cdot)$ of the Crump-Mode-Jagers process (see (3.164)) and hence

$$\mathcal{U}(\alpha^{-1}\log N + t,\cdot,\cdot) \underset{N\to\infty}{\longrightarrow} \mathcal{U}(\infty,\cdot,\cdot), \; a.s.. \tag{3.360}$$

Now take collisions into account and consider the limit of $U^N(\frac{\log N}{\alpha}+t,\cdot,\cdot)$ as $N \to \infty,\; t \to -\infty$. Combining (3.359) and (3.360) it follows that for $\eta > 0$ there exists $\tau(\eta)$ such that

$$\limsup_{N\to\infty} P\left(\sup_{t\leq\tau(\eta)} \|U^N(\frac{\log N}{\alpha}+t),\cdot,\cdot) - \mathcal{U}(\infty,\cdot,\cdot)\| > 2\eta\right) < \eta. \tag{3.361}$$

Therefore

$$\lim_{t\to-\infty} U(t,\cdot,\cdot) = \mathcal{U}(\infty,\cdot,\cdot). \tag{3.362}$$

Finally we have to verify that indeed $(e^{-\alpha t}\frac{\widetilde{u}^N(t)}{N})_{t\in\mathbb{R}}$ converges to a limit $(e^{-\alpha t}u(t))_{t\in\mathbb{R}}$ such that $e^{-\alpha t}u(t) \to W^{k,\ell}$ as $t \to -\infty$, where $W^{k,\ell}$ arises as the growth constant of the collision-free process. This was already essentially done in the proof of (3.311), if we observe that the difference between the collision-free process and $\widetilde{u}^N(t)$ is bounded by the number of black sites, whose intensity vanishes as $t \to -\infty$, since

$$\limsup_{N\to\infty} P[\tau^N(\varepsilon) \leq \alpha^{-1}\log N + t] \to 0 \text{ as } t \to -\infty \tag{3.363}$$

and hence the intensity of black sites is $O(\varepsilon^2)$ at time $\tau^N(\varepsilon)$ for N large.

This completes the proof of part (d).

Part 3: Occupation density of dual particle system

Proof of (e)

Note first that (3.319) follows from what we proved for the collision-free regime, in particular the fact that $\sup(e^{-\alpha t}K_t|t>0)<\infty$.

(3.320) Next note that it suffices to prove the convergence in (3.318) to obtain (3.320) using (3.319). Namely we then have pointwise convergence on a suitable probability space and an upper bound is given by the collision-free process with single sites in equilibrium, which provides an integrable integrand and is independent of N and indeed using Proposition 3.8, part (b) we see that (3.320) holds.

(3.318) Using Proposition 3.8 part (c) we see that for (3.318) we have to reduce the problem concerning $u^N U^N$ and uU to what we showed there for u^N and u respectively U^N and U. Namely each of them converges weakly. However since

$$(3.364) \quad (u(t), U(t,\cdot)) \longrightarrow u(t)U(t,\cdot)$$

is a continuous mapping this follows from the convergence properties we proved for $(N^{-1}\widetilde{u}^N, \widetilde{U}^N)$ as $N\to\infty$.

This completes the proof of Proposition 3.8.

Remark 28 *Consider the normalized first passage times $\bar{\tau}^N(\varepsilon)$, where $\bar{\tau}^N(\varepsilon)$ is the first time where $\Pi_s^{N,k,\ell}$ (rather than $K_s^{N,k,\ell}$) reaches $\lfloor \varepsilon N\rfloor$.*

Then one can show that for every $\varepsilon\in(0,1)$:

$$(3.365) \quad \mathcal{L}[\bar{\tau}^N(\varepsilon) - \frac{\log N}{\alpha}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\bar{\tau}(\varepsilon)],$$

with

$$(3.366) \quad \bar{\tau}(\varepsilon) = \inf(t\in\mathbb{R}|u(t)U(t,\mathbb{R},\mathbb{N})=\varepsilon),$$

and u is the entrance law with $\lim\limits_{t\to\infty} e^{\alpha|t|}u(t)=W$.

3.2.9 Dual population in the transition regime: asymptotic expansion

The purpose of this subsubsection is to use and extend the techniques we developed above to later be able to connect the early times (studied later in Subsection 3.3) and the late times studied in Subsubsection 3.2.8 above.

Return first to our original population model. On the side of the original process the limit in the first time scale, i.e. of times $O(1)$ as $N\to\infty$, is described by the branching process $(\beth_t^m)_{t\geq 0}$ and in the second time range, i.e. after times $\alpha^{-1}\log N$ by the deterministic McKean-Vlasov equation but with random initial condition. For both these limits we have derived (respectively for $\beth_t^m$ we will in Subsection 3.3) exponential growth at the same rate α and for the first we had obtained the growth constant $\mathcal{W}^*$ and for the latter the growth constant ${}^*\mathcal{W}$. The objective below is to show that we can consider the limit simultaneously in both time scales in such a way that the final random value arising from the branching process describing the number of type-2 sites at small times provides the initial value of the McKean-Vlasov equation describing the evolution from emergence on till fixation meaning that in fact ${}^*\mathcal{W}$ and $\mathcal{W}^*$ have the same distribution.

This section provides the analysis of the dual needed for this purpose. The completion of the argument is obtained in the following subsubsection which is devoted to the limiting branching process and the transfer back of properties of the dual to the original process.

Recall the definition of u^N, U^N in (3.235)- (3.239). In this section we deal with the relation of the two basic limits as $N\to\infty$ of the functional of the dual population given by $(u^N(t), U^N(t))$, namely, the limit in the two different time scales

$$(3.367) \quad t_N = o(T_N) \text{ and } \quad t_N = T_N + t, \quad \text{where} \quad T_N := \frac{\log N}{\alpha} \text{ and } t\in\mathbb{R},$$

as well as the *transition* between these two time scales which *separate* in the limit $N \to \infty$.

The main tool developed in this subsection is the analysis of the dual population in this two-time scale context. For that purpose we make use of two techniques: (1) *Nonlinear evolution equations* for the dual particle systems and their enrichments, (2) *couplings* of particle systems to estimate and control the effects of collisions in the dual process by comparing it with collision-free systems, i.e. the dual of the McKean-Vlasov process.

Heuristically we can approximate u^N/N by $\widetilde{u}^N$ which are the solutions of the ODE

$$\frac{d\widetilde{u}^N(t)}{dt} = \alpha_N(t)\widetilde{u}^N(t)(1 - \frac{b_N(t)}{N}\widetilde{u}^N(t)),\ t \geq t_0(N), \tag{3.368}$$

$$b_N(t) = 1 + \frac{\gamma_N(t)}{\alpha_N(t)}, \quad \widetilde{u}^N(t_0(N)) = u^N(t_0(N)) = W_N e^{\alpha t_0(N)}, \tag{3.369}$$

which allows to get some feeling for the behaviour based on explicit calculation for the case where α_N and γ_N are constant and explicit formulas for the solution of the ODE (3.368) and (3.369) can be given.

There are two regimes depending on which of the time scales in (3.367) is used: the first (linear) regime where $N^{-1}u^N(t_N) \to 0$ as $N \to \infty$ together with $\alpha_N(t_N) \to \alpha$ and $\gamma_N(t_N) \to \gamma$ and the second (nonlinear) regime where $u^N(t_N) = O(N)$ and where $\alpha_N(t_N), \gamma_N(t_N)$ differ from α, γ.

We have proved in Subsubsection 3.2.3 that for times $t_N \to -\infty$ the number of sites occupied by the dual process at times $\frac{\log N}{\alpha} + t_N$ and multiplied by N^{-1} converges to zero in probability. On the other hand we saw in Subsubsection 3.2.7 that assuming that in law

$$\frac{1}{N}u^N(\frac{\log N}{\alpha} + t_0) \underset{N\to\infty}{\Longrightarrow} u(t_0), \tag{3.370}$$

then in law on path space

$$\{\frac{1}{N}u^N(\frac{\log N}{\alpha} + t)\}_{t \geq t_0} \underset{N\to\infty}{\Longrightarrow} \{u(t)\}_{t \geq t_0} \tag{3.371}$$

and given the state at time t the limiting dual dynamics $u(t)$ is deterministic and nonlinear, namely, $(u(t), U(t))_{t \geq t_0}$ is for all $t_0 \in \mathbb{R}$ the solution to the system (3.248), (3.249) where in the latter quantity we integrate out the age. Moreover we proved that this system has a unique solution satisfying

$$\lim_{t\to-\infty} e^{\alpha|t|}u(t) = W, \quad \lim_{t\to-\infty} U(t) = \mathcal{U}(\infty), \tag{3.372}$$

where $\mathcal{U}(\infty)$ is the stable size distribution of the CMJ process induced by the collection of the occupied sites of the McKean-Vlasov dual process.

In order to later on relate ${}^*\mathcal{W}$ and $\mathcal{W}^*$ we next focus on ${}^*\mathcal{W}$ and try to get more information on its law. This requires, as we shall see later, when we return to the original process from the dual to obtain in (3.372) the higher order terms as $t \to -\infty$ for the limiting equation (as $N \to \infty$) as well as in the approximation as $N \to \infty$ of this behaviour. This means we want to write for $t \to -\infty$ the limiting (as $N \to \infty$) intensity of $N^{-1}\widetilde{u}^N$, denoted u:

$$u(t) = We^{\alpha t} - Const \cdot W^2 e^{2\alpha t} + o(e^{2\alpha t}) \tag{3.373}$$

and determine the constant, but moreover we want to consider for $N \to \infty, t \to -\infty$ the quantity

$$u^N(\frac{\log N}{\alpha} + t) = W_N(t)e^{\alpha t} \cdot N + C(t, N) \tag{3.374}$$

and to estimate the order in both N and t of the correction term $C(t, N)$. The latter is equivalent to determining up to which order the expansion in (3.373) is approximated by the finite N-system.

First we turn to the question in (3.374) and then to (3.373) for each point of view formulating a separate proposition.

We therefore refine (3.372) by providing speed of convergence, in t and uniformity, in N, results.

Proposition 3.9 *(Order of approximation of entrance law in N and t)*

(a) Let $T_N = \frac{\log N}{\alpha}$ and $0 < \delta < 1$. Then $(u^N(t), U^N(t))$ and $(u(t), U(t))$ have the following three properties:

We can couple the $(u^N(t), U^N(t))_{t\geq 0}$ with a fixed CMJ process $(K_t, \mathcal{U}(t))_{t\geq 0}$ such that

$$e^{-\alpha t} K_t \longrightarrow W \text{ a.s. as } t \to \infty, \tag{3.375}$$

$$\lim_{t\to-\infty} \limsup_{N\to\infty} P\left(|e^{-\alpha t}\frac{1}{N} u^N(T_N + t) - W| > e^{\delta\alpha t}\right) = 0 \tag{3.376}$$

and

$$\lim_{t\to-\infty} \lim_{N\to\infty} \left(\frac{e^{-\alpha t}}{N} u^N(T_N + t), U^N(T_N + t)\right) = (W, \mathcal{U}(\infty)). \tag{3.377}$$

(b) Given W the system (3.248), (3.249) has unique solution $(u(t), U(t))$ with $e^{-\alpha t}u(t) \to W$ as $t \to -\infty$ which can be realized together with $\{(u^N, U^N), N \in \mathbb{N}\}$ on one probability space such that the process

$$(e^{-\alpha t} N^{-1} u^N(T_N + t))_{t\in\mathbb{R}} \tag{3.378}$$

can for every $\delta \in (0, 1)$ be approximated by $(e^{-\alpha t}u(t))_{t\in\mathbb{R}}$ with "initial condition" W, up to order $e^{\alpha\delta t}$ for $t \in \mathbb{R}$ uniformly in N. $\square$

We now have established a precise relation between (u^N, U^N) and the limiting entrance law (u, U). We now return to the solution (u, U) to (3.248), (3.249) and $\alpha(t)$, $\gamma(t)$ as defined in equation (3.246). The purpose of the rest of this section is to identify the asymptotic behaviour of this nonlinear system as $t \to -\infty$ up to higher order terms, i.e. we want to identify in particular the correction of order $e^{2\alpha t}$ as $t \to -\infty$ which is due to the occurrence of collisions by migration of the dual individuals which by subsequent coalescence of the collided individuals can change the behaviour of the limiting system.

Proposition 3.10 *(Transition between linear and nonlinear regime limit dynamics (u, U))*

(a) The pair (u, U) and the functionals α and γ arising as the limit in (3.312)-(3.314) via (3.246) satisfy (here we suppress the k, ℓ):

$$(\alpha(t) + \gamma(t))u(t) \leq (\alpha + \gamma)W e^{\alpha t}, \tag{3.379}$$

$$\lim_{t\to-\infty} \alpha(t) = \alpha, \quad \lim_{t\to-\infty} U(t) = \mathcal{U}(\infty), \quad \lim_{t\to-\infty} \gamma(t) = \gamma, \tag{3.380}$$

$$u(t) = W e^{\alpha t} - \kappa W^2 e^{2\alpha t} + O(e^{3\alpha t}) \quad \text{as } t \to -\infty, \text{ for some constant } \kappa > 0 \tag{3.381}$$

and furthermore as $t \to -\infty$ the functions $\alpha(\cdot)$ and $U(\cdot)$ satisfy that

$$\alpha(t) = \alpha + \widetilde{\alpha}_0 e^{\alpha t} + O(e^{2\alpha t}), \quad U(t) = \mathcal{U}(\infty) + \widetilde{U}_0 e^{\alpha t} + O(e^{2\alpha t}), \tag{3.382}$$

$$\gamma(t) = \gamma + \widetilde{\gamma}_0 e^{\alpha t} + O(e^{2\alpha t}), \tag{3.383}$$

where $\widetilde{\alpha}_0$ is a positive number $\widetilde{U}_0$ is an $\mathbb{N}$-vector and $\widetilde{\gamma}_0 = \widetilde{U}_0(1)$.

Using a multicolor system construction, we will see below that all the randomness is given by the (non-degenerate positive) random variable W, namely

(3.384) $\widetilde{\alpha}_0$ *is explicitly given by (3.569), as* $Const_1 \cdot W$,

(3.385) $\widetilde{U}_0$ *is explicitly given by (3.575) and has the form* $W \cdot \vec{const}, \widetilde{\gamma}_0 = Const_2 \cdot W$.

(b) The total number of particles satisfies

(3.386) $(\alpha(t) + \gamma(t))u(t) = (\alpha + \gamma)We^{\alpha t} - \kappa^* W^2 e^{2\alpha t} + O(e^{3\alpha t}) \quad$ *as* $t \to -\infty$,

for some constant $\kappa^* > 0$.

More precisely with

(3.387) $\kappa^* W^2 = (\widetilde{\alpha}_0 + \widetilde{\gamma}_0)W + \kappa W^2 = W^2(Const_1 + Const_2 + \kappa)$,

we have that

(3.388) $\limsup_{t\to-\infty} |(\alpha(t) + \gamma(t))u(t) - (\alpha + \gamma)We^{\alpha t} - \kappa^* W^2 e^{2\alpha t}| \cdot e^{-3\alpha t} < \infty.$

(c) Furthermore $U(t, \cdot, \cdot)$ *is uniformly continuous at* $t = -\infty$ *and as* $t \to -\infty$

(3.389) $\|U(t) - \mathcal{U}(\infty) - \widetilde{U}_0 e^{\alpha t}\|_{\text{var}} = O(e^{2\alpha t}).$ □

Remark 29 *Note that the results above show that all the relevant randomness in the dual process sits indeed in the random variable W.*

The proof proceeds in four steps. The proof of Propositions 3.9 and 3.10 are based on an enriched version of the dual particle system η and a reformulation of the nonlinear system (3.248) and (3.249) which are given in Steps 1 and 2 below which are then followed by the Steps 3 and 4 giving the proof of the two propositions using these tools. In Step 1 we focus on the finite N system in a time regime where collisions due to migration become essential, while the Step 2 develops the analysis of the limiting system as $N \to \infty$ in this time regime. The main focus here is on the behaviour as the effect of collisions becomes small and to investigate its precise asymptotics in this regime.

Step 1: Modified and enriched coloured particle system

Our goal is to realize on one probability space the CMJ-process and the dual particle system in such a way that we can identify in a simple way the difference between the two dynamics with a higher degree of accuracy than before by using three colours white, black and red. In particular we can identify the part of the population involved in one or in more than one collision (more than two etc.). This joint probability space can be explicitly constructed easily, based on collections of Poisson processes. Since no measure theoretic subtleties occur, we do not write out this construction in all its lengthy detail and just spell out the evolution rules.

In order to identify the contributions at time $\alpha^{-1}\log N + t$ of order $e^{\alpha t}, e^{2\alpha t}, e^{3\alpha t}$, we want to expand the number of occupied sites at time $\alpha^{-1}\log N + t$, using quantities $W_N(t)$ defined as $e^{-\alpha t}K_t^N$ and then analog higher-order objects, in the form:

(3.390) $u^N(T_N + t) \sim W_N(t)e^{\alpha t} - W'_N(t)e^{2\alpha t} + W''_N(t)e^{3\alpha t}$,

with $T_N = \alpha^{-1}\log N$ and $N^{-1}W_N(t), N^{-2}W'_N(t), N^{-3}W''_N(t)$ converging as $N \to \infty, t \to -\infty$, more precisely,

$$(3.391) \quad \begin{array}{l} \lim_{t\to-\infty}\lim_{N\to\infty}[e^{-\alpha t}N^{-1}u_N(T_N + t)] = W \\ \lim_{t\to-\infty}\lim_{N\to\infty}[e^{-2\alpha t}N^{-2}[(WNe^{\alpha t} - u^N(T_N + t))]i] = \kappa W^2 \quad , \\ \lim_{t\to-\infty}\lim_{N\to\infty}[e^{3\alpha t}N^{-3}[u_N(T_N + t)) - WNe^{\alpha t} - \kappa N^2 W^2 e^{2\alpha t}]] < \infty, \end{array}$$

where the limits are needed in probability and in L^1. The latter convergence follows from the convergence in probability and the finiteness of the moments of W (see Lemma 3.4).

We use the dual particle system to obtain information on the asymptotics as $t \to -\infty$ of the pair (u, U) and also to identify their random initial condition. However to carry this out we must enrich the *dual particle system* to a *multicolour* particle system on an *enriched geographic space*.

Since we want to read off more properties than in the emergence argument based on the mean where we had white, black and red particles we shall need now some more colours which allow us to record particles which have been involved in one, two, etc. collisions and with associated colours their counterpart in a collision-free system.

Furthermore in order to realize the CMJ and the dual on one probability space we have to couple the evolution of certain colours where one belongs to the CMJ-part and the other to the dual particle system part. This will also require us to refine the geographic space $\{1, \cdots, N\} + \mathbb{N}$ (where the black particles were collision-free and moving on $\mathbb{N}$) we had before for the white, red, black particles system by adding a further copy $\mathbb{N}$ (or more if we need more accuracy).

More precisely we now introduce a *multicolour particle system* that is obtained by modifying and enriching the coloured particle system defined by (3.177) - (3.182) by introducing (1) new colours and (2) new type of sites. The number of new colours and sites necessary depends on how precisely we want to control the behaviour as $t \to -\infty$ meaning up to which higher order terms as $t \to -\infty$ we want to go.

We first need a new colour (green) which allows us to control exactly (i.e. not just estimating from above) the difference between the dual particle system η and our old white-black collision-free particle system. Recall that in the old system we created a red-black pair upon a collision such that the further evolution of black remains collision-free and the red follows the true mechanism of the dual particle system. A real difference in the total number of particles between dual and CMJ will however occur only if the red particle coalesces with a white particle but no such loss of an individual occurs in the CMJ-process which now will have one particle more due to this transition. To mark this we will use the colour *green* to mark the dissappearing red particle in the dual.

To allow a fine comparison between the white-red and the white-black system we will use coupling techniques which allow us to estimate not only the effect of *collisions* (and subsequent coalescents) but also *recollisions*, the latter will generate parallel to the red-black construction two new colours *purple* (in the dual system) and *blue* (in the collision-free comparison system). For this purpose we have to modify the evolution rules when red particles collide or red particles collide with white particles. For the bookkeeping of the effects from these events we use the new further colours.

The white particles in the old process also have the same dynamics as the white particles in our new formulation but in addition to white, red, and black particles the *green, purple* and *blue* particles can have now locations in

$$\{1, 2, \cdots, N\} + \mathbb{N} + \mathbb{N}, \tag{3.392}$$

which we will explain as we describe the evolution rules.

In order to achieve all these goals of a higher order expansion the modified dynamics should satisfy the following requirements:

- for each N the union of black plus white particles is equivalent to the particle system without collisions and can be defined on a common probability space by a single CMJ process,
- the white plus red plus purple particles give a version of $(u^N(t), U^N(t, \cdot, \cdot))$,
- the red plus purple particles give a refinement of the red particle system in the white-black-red process used earlier, i.e. *both together* have the same dynamics as the set of red particles in (3.2.6),

- the green particles describe the loss due to coalescence after collision, i.e. after a particle (white, red or purple) migrated to an occupied site on $\{1, \cdots, N\}$ (by a white, red or purple particles) and then coalesced with such particle at this site,
- the black particles are placed and evolve on the second copy of $\mathbb{N}$ and blue particles are placed and evolve on the first copy of $\mathbb{N}$.
- We can match certain pairs of red and black respectively purple and blue particles when they are created to better compare the CMJ-part and the one with the dual dynamics.
- The number of the sites occupied by red, green and blue particles has the same law as the number of sites occupied by black particles.

For each N the evolution rules below define a *Markov pure jump process* which describes a growing population of individuals carrying a *location* and a *colour* and in addition the information which pairs of individuals are *coupled*.

Let I_n be a copy of $\{1, \cdots, n\}$. Denote by

(3.393) $\quad C = \{$ white, black, red, purple, blue, green$\}$,

Then the state space is given by the union over n of the set of maps

(3.394) $\quad I_n \longrightarrow C \times (\{1, \cdots, N\} + \mathbb{N} + \mathbb{N}\}) \times (I_n^2).$

If we just count the number particles with a certain colour and location we get again a Markov pure jump process, specifying a multitype particle system on $\{1, 2, \ldots, N\} + \mathbb{N} + \mathbb{N}$. The state space of this system is given by:

(3.395) $\quad (\mathbb{N}_0)^{(\{\{1,2,\ldots,N\}+\mathbb{N}+\mathbb{N}\}\times C)}.$

We shall work with both processes.

In order to satisfy the properties specified above, the dynamics of the enriched coloured particle systems has the following dynamics:

- Given that k is the number of sites in $\{1, 2, \ldots, N\}$ which are occupied, and a white particle migrates to another site in $\{1, \cdots, N\}$ the outcome is given as in (3.179), (3.180) and (3.181) (a *red-black pair* is created) but we record that these particles form a red-black pair. White particles give birth to white particles and two white particles coalesce to produce one white particle.
- Black particles evolve as before but live on the second copy of $\mathbb{N}$.
- Red particles give birth to red particles, two red particles at the same site coalesce to one *red particle*.
- When a red particle coalesces with a white particle (at the same site) the outcome is a white particle and a *green* particle at this site. Moreover we then couple the green particle with the black particle that had been associated with the involved red particle.
- Given that the current number of sites occupied by white, red, purple or green particles on $\{1, \cdots, N\}$ is ℓ, when a red particle migrates it moves to an empty site in $\{1, \ldots, N\}$ with probability $1 - \frac{\ell}{N}$ and with probability $\frac{\ell}{N}$ it dies and produces a *blue-purple pair* where the blue particle is placed at the first empty site in in the first copy of $\mathbb{N}$ and the purple particle is placed at a randomly chosen occupied site on $\{1, 2, \cdots, N\}$. We also associate the blue particle with the black particle that had been associated with the involved red particle.

- Blue particles have the same dynamics as black particles.
- Purple particles have the same dynamics as the dual particle systems η on $\{1, \dots, N\}$, in particular they migrate to a randomly chosen site in $\{1, \dots, N\}$.
- Green particles give birth to green particles and two green particles at one site coalesce giving one green particle. Green particles on $\{1, 2, \cdots, N\}$ migrate to empty sites on $\{1, \cdots, N\}$ with probability $1 - \frac{\ell}{N}$ and with probability $\frac{\ell}{N}$ to the first empty site in the first copy of $\mathbb{N}$ where ℓ is the number of occupied sites on $\{1, \cdots, N\}$. Green particles on the first copy of $\mathbb{N}$ migrate to the first empty site.
- When a green and red particle coalesce they produce a just red particle.

  Remark 30 *The reason for this rule is as follows. The red and green pair correspond to two red particles one of which has coalesced with a white particle. The green particle corresponds to a particle loss due to collision and coalescence. First note that the green particle which coalesces with a red one must have been created at that site. Hence both red and green must descend from the same collision of a particle arriving at the site and hence the coupled black particles share the same site. The red and green particle are coupled with a black or blue particle each. The two black particles can coalesce only if they descend from the same black particle after arriving at the current site. In this case the number of red and green, respectively black and blue is reduced both by one and the original loss of black-white versus white-red is cancelled. When the green and red coalesce this corresponds to the coalescence of two red particles that would have been at a site having no white particles. Then the coalescence reduces the number of red particles to one. This means that the potential loss is canceled out and in both cases we end up with a single red particle.*

  Green and white particles do not coalesce.
- We couple for a newly created red-black pair the birth, coalescence with offspring up to the time of the first collision of a red particle with a site occupied by red or white particles. We then continue by coupling for all future times the associated black particle with the blue particle (from the blue-purple pair) created at the collision time.
- When red and purple particles coalesce they produce a red particle.
- When a purple particle coalesces with a white particle the outcome is a white and a green particle at the site.

Note that these rules lead to a system with the six desired properties we had listed below (3.392).

The state space of this system is given by:

$$(\{\{1, 2, \dots, N\} + \mathbb{N} + \mathbb{N}\} \times C)^{\mathbb{N}_0}. \tag{3.396}$$

We shall single out specific subsystems comprised of particles of certain colours, for example, the *WRP-system* (white, red, purple) or the *WRGB-system* (white, red, green, blue) which have state-spaces:

$$\mathbb{N}_0^{(\{\{1,2,\dots,N\}+\mathbb{N}+\mathbb{N}\}\times\{W,R,P,\})}, \quad \mathbb{N}_0^{(\{\{1,2,\dots,N\}+\mathbb{N}+\mathbb{N}\}\times\{W,R,G,B\})}, \tag{3.397}$$

respectively. If we need more information, then we have the analogue of (3.394).

Observe that white, red, and purple particles are located only in $\{1, \dots, N\}$, blue particles are located only in the first copy of $\mathbb{N}$ and green particles can be in either $\{1, \dots, N\}$ or first copy of $\mathbb{N}$

and all black particles live in the second copy of $\mathbb{N}$. It is important to note that the transitions of green particles do not depend on whether they are located in $\{1, \cdots, N\}$ or in the first copy of $\mathbb{N}$.

We note that the difference between the number of red plus purple particles and the number of black particles is due to the coalescence of red or purple particles with white particles. Each such loss is compensated by the creation of a green particle and we note that green particles never migrate to an occupied site.

By the construction above we define on a common probability space five *coupled* particle systems,

- a CMJ-process which is the union of black and white particles which give a version of $(K_t, U(t))_{t\geq 0}$,
- the white, red and purple particles generate a version of $(u^N(t), U^N(t))$,
- a CMJ process which is the union of white, red, green and blue particles and which generates a version of the pair $(K_t, U(t))_{t\geq 0}$,
- a subset of this above system, the white system generating the pair $(K^N_{(W),t}, U^N_{(W)}(t, \cdot))$, a CMJ-process which is the union of black and white particles which give a version of $(K_t, U(t))_{t\geq 0}$,
- a *coupled* subsystem, on the one hand the system of red and purple and on the other hand a subset of the black particles. This coupling is induced by the convention how purple-blue and red-black pairs evolve.

As before in the analysis of (u, U) it suffices for our purposes to study a functional of the multicolour particle system by observing at sites only how many particles of the various colours occur and counting the number of sites of a specific colour configuration. Our functional of a state is given by the counting process

$$\Psi^N_t(i_W, i_R, i_P, i_G, i_B, i_{BL}), \tag{3.398}$$

denoting the number of sites containing i_W white, i_R red, i_P purple, i_G green, i_B blue and i_{BL} black particles at time t.

The process $(\Psi^N_t)_{t\geq 0}$ evolves itself as a *pure jump-strong Markov process* with state space the counting measures on $C^{\mathbb{N}}$, the set of colour configurations, due to the fact that the dynamic only depends on the vector of the numbers of particles of the various colours at a site. Similarly we define pure Markov jump processes

$$\Psi^N_{W,t}(i_W), \quad \Psi^N_{WRP,t}(i_W, i_R, i_P), \quad \Psi^N_{WRGB,t}(i_W, i_R, i_G, i_B). \tag{3.399}$$

Next we want to define the coloured versions of the (u, U) system and of certain subsystems. We therefore define the number of sites exhibiting certain colours:

$$K_t = \sum_{i_i, \cdots, i_{BL}} 1_{i_W + i_{BL} \geq 1} \Psi^N_t(i_W, i_R, i_P, i_G, i_B, i_{BL}), \tag{3.400}$$

$$K^N_{WRP,t} = \sum_{i_W, i_R, i_P} 1_{i_W + i_R + i_P \geq 1} \Psi^N_{WRP,t}(i_W, i_R, i_P), \tag{3.401}$$

$$K^N_{WRGB,t} = \sum_{i_W, i_R, i_G, i_B} 1_{i_W + i_R + i_G + i_B \geq 1} \Psi^N_{WRGB,t}(i_W, i_R, i_G, i_B) \tag{3.402}$$

and occasionally we make use of

$$K^N_{C,t} = \sum_{i_L, \cdot L \in C} 1_{\sum\limits_{L\in C} i_L \geq 1} \Psi^N_{C,t}(i_C), \quad C \subseteq \{W, R, P, G, B, BL\}, \quad i_C = (i_L)_{L \in C}. \tag{3.403}$$

Now we can define the coloured relatives of (u^N, U^N) due to our construction all on one probability space, setting

$$(u(t), u^N_W(t), u^N_{WRP}(t), u^N_{WRGB}(t)) = (K_t, K^N_{W,t}, K^N_{WRP,t}, K^N_{WRGB,t}), \tag{3.404}$$

$$\begin{aligned} &U^N_{W,t}(i_W) = \frac{\Psi^N_{W,t}(i_W)}{K^N_{W,t}}, \\ &U^N_{WRP,t}(i_W, i_R, i_P) = \frac{\Psi^N_{WRP,t}(i_W,i_R,i_P)}{K^N_{WRP,t}}, \\ &U^N_{WRGB,t}(i_W, i_R, i_G, i_B) = \frac{\Psi^N_{WRGB,t}(i_W,i_R,i_G,i_B)}{K^N_{WRGB,t}}. \end{aligned} \tag{3.405}$$

Lemma 3.11 *(Properties of enriched coloured particle system)*

(a) Ignoring colour the process $(u^N_{WRP,t}, U^N_{WRP}(t,\cdot))_{t\geq 0}$ has the same law as $((u^N_t, U^N_t))_{t\geq 0}$ given by (3.235)- (3.239) .

(b) Ignoring colour, the the total number of particles or occupied sites in the WRGB-system has the same distribution as the number of particles respectively occupied sites in W-BL-system and therefore we can identify the total number of occupied sites with the total number of individuals of a CMJ process which we denote by $(K_t)_{t\geq 0}$.

(c) The total number of particles (or sites) in the WRP-system is less than that in the WRB-system and larger that that in the WR-system a.s.

(d) Consider the coloured particle system at times

$$T_N + t, \quad T_N = \alpha^{-1} \log N. \tag{3.406}$$

The total number of purple or blue particles is $O(e^{3\alpha t})$, more precisely, we have uniformly in N,

$$\begin{aligned} &\sum_{i_W, i_R, i_P} i_P \cdot \Psi^N_{WRP, T_N+t}(i_W, i_R, i_P) \\ &\leq \sum_{i_W, i_R, i_G, i_B} i_B \cdot \Psi^N_{WRGB, T_N+t}(i_W, i_R, i_G, i_B) \leq const \cdot N W_* e^{3\alpha t}, \end{aligned} \tag{3.407}$$

where

$$W_* = \sup_t (e^{-\alpha t} K_t) < \infty, \text{ a.s. } . \tag{3.408}$$

(e) We consider again a time horizon as in (d). Define $W_N(t) = N^{-1} e^{-\alpha t} u^N(T_N + t)$. The process counting the number of red particles produced by collisions of white particles is bounded above by an inhomogeneous Poisson process with rate

$$N e^{2\alpha t} (W^2_N(t))^2 \tag{3.409}$$

where

$$\lim_{N\to\infty} W_N(t) = W \text{ for each } t \in \mathbb{R}. \qquad \square \tag{3.410}$$

Corollary 3.12 *The normalized (by N^{-1}) number of red respectively purple particles at time T_N+t conditioned on (W, W_*) is uniformly in N*

$$O(e^{2\alpha t}), \text{ respectively } O(e^{3\alpha t}), \text{ as } t \to -\infty. \qquad \square \tag{3.411}$$

Remark 31 *By introducing further colours, for example, following the first collision of a purple particle with a site occupied by some other colour introducing a new pair of colours analoguous to purple and blue, etc., we could obtain upper and lower bounds with errors of order $O(e^{k\alpha t})$ for any $k \in \mathbb{N}$.*

Proof of Lemma 3.11 (a) follows by observing that combined number of white, red and purple particles at a site behaves exactly like the dual particle process of typical site and when they migrate they can have collisions according to the same rules as the dual particle system. This follows since when a white particle migrates it collides with the correct probability and then becomes red and when a red particle migrates it can collide and become purple and when a purple particle migrates it can collide again following the same rule (and remains purple).

(b) Since the green particles compensate for the loss of red particles due to coalescence after collision of a white with another white particle and the blue particles arise upon collision of red with red or white and they evolve as the black particles, the total number of particles is as in the W-BL system.

(c) follows since (i) the blue particles are produced in one to one correspondence with the purple particles and (ii) the purple particles can suffer loss due to collision and coalescence but this does not occur with the blue particles.

(d), (e) We have proved earlier in this section using the CMJ-theory that the normalized rate for the number of white particles

$$\frac{1}{N} u^N_W(T_N + t) = O(e^{\alpha t}), \tag{3.412}$$

if we condition on $\{W, W_*\}$ since for all N this number is smaller than the number of the white and black particles together, which is then asymptotically $W_N(t) N e^{\alpha t}$, with $W_N(t) \to W$ as $N \to \infty$ by the CMJ-theory. When white particles migrate they collide with another white particle with probability $u^N_W(T_N + t)/N$. Moreover white migrants are at times before $T_N + t$ produced at rate at most $W_N(t)\ N\alpha e^{\alpha t}$. Therefore the normalized rate for the number of red particles produced conditioned on $\{W, W_*\}$ is at time $T_N + t$ is satisfying $O(Ne^{2\alpha t})$.

Finally a purple or blue particle occurs when a red particle collides with a white or red and therefore the rate of this event is of order

$$O(Ne^{2\alpha t}) \times O(e^{\alpha t}) = O(Ne^{3\alpha t}), \tag{3.413}$$

if we condition again on $\{W, W_*\}$.

This completes the proof of Lemma 3.11 which will be a key tool in the further analysis of the dual process.

Remark 32 *Since the WRGB system can be identified with the W-BL(by assigning the union of the red and green particles at a white site to an empty site in the first copy of $\mathbb{N}$) a system we will primarily work with the former and ignore the BL system. We then have both upper and lower bounds for the WRP system given by the WRB, WR systems, respectively and this will be our primary object for the analysis of the $t \to -\infty$ asymptotics. This will provide upper and lower bounds with error of $O(e^{3\alpha t})$ and therefore determine the second order asymptotics as $t \to -\infty$.*

Step 2: Reformulation of the nonlinear $(u, U), (u_{(C)}, U_{(C)})$ equations

This step has two parts, first rewriting the (u, U) equation and then a second part where we do this for the multicolour version of this equation.

Part 1 We will start by bringing the equations of the limit dynamic (u, U) in a form suitable for the purpose of the proof of Propositions 3.9 and 3.10. Recall that (u, U) solves the following system of differential equations in the Banach space $(\mathbb{R} \times L_1(\mathbb{N}, \nu), \|\cdot\|)$ (remember for the first line that U is a normalized quantity):

$$\frac{du(t)}{dt} = \alpha(t)(1-u(t))u(t) - \gamma(t)u^2(t), \quad t \geq t_0, \tag{3.414}$$

$$\begin{aligned} \frac{\partial U(t,j)}{\partial t} = \quad & +s(j-1)1_{j\neq 1}U(t,j-1) - sjU(t,j) \\ & +\frac{d}{2}(j+1)jU(t,j+1) - \frac{d}{2}j(j-1))1_{j\neq 1}U(t,j) \\ & +c(j+1)U(t,j+1) - cjU(t,j)1_{j\neq 1} \\ & +\alpha(t)1_{j=1} \\ & +u(t)(\alpha(t)+\gamma(t))[U(t,j-1)1_{j\neq 1} - U(t,j)]) \\ & -u(t)\left(\alpha(t)+\gamma(t)\right)1_{j=1} \\ & -\Big(\alpha(t)(1-u(t)) - \gamma(t)u(t)\Big)\cdot U(t,j). \end{aligned} \tag{3.415}$$

Remark 33 *We want to interpret these evolution equations (3.414),(3.415) by a particle system of the type of the mean-field dual but now making more explicit the role of collision and in particular the first collisions of particles. This will allow us to analyse the behaviour as $t \to -\infty$ of the nonlinear evolution equation above. The analysis will be based on the fact that $u(t)$ arises as the limit of $N^{-1}u^N(T_N+t)$ and similarly $U(t)$ as the limit of $U^N(T_N+t)$ where $T_N = \alpha^{-1}\log N$. This will allow us in the second part of this Step 2 to introduce enrichments of the solution of the nonlinear equations through coloured particle systems. Then by taking the $N \to \infty$ limit of the normalized coloured particle systems we obtain a coloured limiting evolution and by that information about the nonlinear original system.*

We next rewrite the equation (3.415) in a form suitable for the analysis of the solution viewed as a *perturbation* of the linear (collision-free and hence $u \equiv 0$, $\alpha(t) \equiv \alpha, \gamma(t) = \gamma$) equation in the limit as $t \to -\infty$. We shall see below in Lemma 3.14, that with $u(t) \equiv 0$, the system (3.415) has the stable age distribution $\mathcal{U}(\infty, \mathbb{R}, \cdot)$ of the CMJ as equilibrium and $\alpha = c\sum_{k=2}^{\infty} k\mathcal{U}(\infty, \mathbb{R}, k)$. Hence we should organize the r.h.s. of (3.415) in such a way that we isolate the linear part on the one hand and the nonlinear perturbations of this linear part on the other hand.

We define for every parameter $a \in (0,\infty)$ (for which we shall later choose the value α) the triple of $\mathbb{N}\times\mathbb{N}$-matrices

$$(Q_0^a, Q_1, L), \tag{3.416}$$

by the equations:

$$Q_0^a = Q_0^0 - aI + a1_{j=1} = \left(q_{j,k}\right)_{j,k\in\mathbb{N}} \left(= (q_{j,k}(a))_{j,k\in\mathbb{N}}\right), \tag{3.417}$$

where

$$\begin{aligned} & q_{12} = s, \quad q_{1,1} = -s \\ & q_{2,3} = 2s, \quad q_{2,1} = d + 2c + a, \\ & q_{j,j+1} = sj, \quad q_{j,j-1} = \frac{d}{2}j(j-1) + cj, \quad q_{j1} = a, \quad j \neq 1,2 \\ & q_{jj} = -sj - cj - \frac{d}{2}j(j-1) - a, \quad j \neq 1, \end{aligned}$$

$$Q_1 = (\widetilde{q}_{jk})_{j,k\in\mathbb{N}}, \text{ where} \quad \widetilde{q}_{jj} = -1, \quad \widetilde{q}_{jk} = 0 \quad j \neq k, \tag{3.418}$$

and finally

$$L = (\ell_{jk}), \quad \ell_{jj} = 0, \text{and for } j \neq 1, \ \ell_{j-1,j} = 1. \tag{3.419}$$

Note that the matrix Q_0^a is for the forward equation but we consider $\bar{Q}_0^a$ for the backward equation. Then for each $a > 0$:

$$\bar{Q}_0^a \text{ generates a semigroup } (S^a(t))_{t\geq 0} \text{ on } L_\infty(\mathbb{N}) \tag{3.420}$$

corresponding to a unique (pure jump) Markov process on $\mathbb{N}$. This process is as follows. The matrix Q_0^a defines a Markov process on $\mathbb{N}$. For $a = 0$ we obtain the birth and death process which corresponds to a colony where we have birth rate s for each particle, coalescence of two particles at rate d and emigration of a particle at rate c, except when there is only one particle. For positive a the process is put at rate a in the state with only one particle.

Furthermore we abbreviate

$$\widetilde{\alpha}(t) = c\sum_{k=2}^{\infty} kU(t,k) - \alpha. \tag{3.421}$$

With these four ingredients equation (3.415) finally reads:

Lemma 3.13 *(Rewritten U-equation)*

$$\frac{\partial U(t)}{\partial t} = U(t)Q_0^\alpha + \widetilde{\alpha}(t)[U(t)Q_1 + 1_{j=1}] + u(t)(\alpha(t)+\gamma(t))[U(t)L - 1_{j=1}]. \quad \square \tag{3.422}$$

Remark 34 *Note that our equations for (u,U) are forward equations and hence all the operators Q_0^a, Q_1, L act "from the right" on $U(t)$.*

Remark 35 *Note that we can consider the equation (3.422) also for arbitrary values $a \in (0,\infty)$. Denote the solutions as*

$$(U^a(t))_{t\in\mathbb{R}}. \tag{3.423}$$

We shall see below how we can characterize U among those by a self-consistency property.

We can later use the following information about the semigroup S^a to characterize the growth rate α of $(u(t))_t \in \mathbb{R}$ in the entrance law (u,U) as $t \to -\infty$.

Lemma 3.14 *(Representation of α)*

Consider the evolution equation for $U^a(t)$ in the regime in which $(u(t))_{t\in\mathbb{R}}$ is identically zero and $\alpha(t) \equiv a$. Then for given $a > 0$ there is a unique equilibrium state (positive eigenvector)

$$\{q_j^*(a)\}_{j\in\mathbb{N}} \text{ for the operator } Q_0^a(\text{ equalling } Q_0^0 - aI + a1_{j=1}). \tag{3.424}$$

Then α is uniquely determined as the fixed point defined by the self-consistency equation

$$\alpha = c\sum_{j=2}^{\infty} jq_j^*(\alpha). \tag{3.425}$$

Also,

$$\{q_j^*(\alpha), j = 1,2,\cdots\} = \{\mathcal{U}(\infty,\mathbb{R},j), j = 1,2,\cdots\} = (p_1,p_2,\ldots), \tag{3.426}$$

the r.h.s. being the stable size distribution of the CMJ process. $\square$

Proof The existence of the unique equilibrium $q_j^*(a)$ follows from standard Markov chain theory, the question is whether the self-consistency equation (3.425) has a solution. Note for this purpose first that $q_j^*(0)$ is the equilibrium for the single site birth and death process appearing in the McKean-Vlasov dual process for one component and observe that $\sum_{j=2}^{\infty} j q_j^*(0) > 0$. We define:

$$F : a \longrightarrow c \sum_{j=2}^{\infty} j q_j^*(a). \tag{3.427}$$

Then we are left to show that the fixed point equation $\alpha = F(\alpha)$ has a solution. We saw above $F(0) > 0$. We claim next that the function is monotone decreasing in a, converging to 0 as $a \to \infty$ and continuous.

To verify the monotonicity, let $a_2 > a_1 > 0$. Recall that the chain for $a \equiv 0$ starting from 1 is stochastically increasing to its equilibrium. Now consider the two Markov chains for a_1, a_2 as a sequence of independent excursions away from 1 which end when a jump to 1 at rate a occurs but otherwise follow the Q_0^0 dynamic. To compare the average height over the excursions, we consider two such excursion lengths given by coupled exponentials $(a_2)^{-1}\mathcal{E}$, $(a_1)^{-1}\mathcal{E}$, respectively. Then we observe by a simple coupling argument that the heights of the a_1-excursion at times $(a_2)^{-1}\mathcal{E} < t \leq (a_1)^{-1}\mathcal{E}$ stochastically dominate the height at time $(a_2)^{-1}\mathcal{E}$. Therefore the average height over an excursion for the a_1-chain is stochastically greater than or equal to that for the a_2-chain.

The continuity follows by noting that the a_2-excursions from zero converge to the a_1 excursion from 0 if $a_2 \downarrow a_1$.

Noting finally that $\sum_{j=2}^{\infty} j q_j^*(a)$ converges to 0 when $a \to \infty$, we obtain the existence and uniqueness of a solution of (3.425) which we read as a fixed point of the equation $\alpha = F(\alpha)$.

Part 2. We now turn to the second part of Step 2, where we give the limiting dynamics of the multicolour enrichment as represented by $(u^N_{(WRPGB)}, U^N_{(WRPGB)})$. We also need systems of other subsets of colours in our arguments and therefore we denote the limiting objects analog to the noncoloured system by

$$(u_{(C)}, U_{(C)}), \tag{3.428}$$

with

$$C = \{W, R, P, G, B\} \text{ or some other colour subset of } \{W, R, P, G, BL, B\}, \tag{3.429}$$

where u_C is a positive real number and $U_{(C)}$ is a measure on

$$(\mathbb{N}_0)^C. \tag{3.430}$$

This set-up means that we consider the processes of *occupied sites*, occupied with the various colour combinations. The dynamics, in the $N \to \infty$ limit, of the WRP-system (or the WRGB-system) at time $T_N + t$ is given by an enrichment $(u_{(WRP)}(t), U_{(WRP)}(t))$ of the nonlinear system (3.414), (3.415). Namely the latter is recovered as follows:

$$u(t) = u_{(WRP)}(t), \quad U(t,k) = \sum_{\{(i_W, i_R, i_P): i_W + i_R + i_P = k\}} U_{(WRP)}(t; i_W, i_R, i_P). \tag{3.431}$$

In order to write down the evolution equation for the quantities from (3.427), we need the changes occuring in the underlying multicolour system and its various subsystems which then induce the changes in the measure on these configurations. For this purpose we need operators associated with the various possible transitions, their rate and their form which are associated with

a particular current state. This state is given by a tuple of the form $(i_W, i_R, i_P, i_B, i_G, i_{BL})$ or a tuple for a smaller set of colours which gives the number of particles of the various colours at a site. Therefore transitions and transition rates are specified by matrices of the form

$$Q_{\underline{i},\underline{j}} \quad , \quad \text{with } \underline{i}, \underline{j} \in (\mathbb{N}_0)^C \quad , \quad C \subseteq \{W, R, P, B, G, BL\}. \tag{3.432}$$

We can now distinguish two different groups of transitions those which concern only particles of *one colour* and then there are transitions where particles of different *colours interact*. We specify these transitions and the corresponding operators in (3.433) and then in (3.435), (3.437).

For the first type the one colour operators let

$$\begin{aligned} Q^{0,i} \quad = \quad & \text{operator } Q^0_0 \text{ applied to the ith colour occupation numbers - see (3.417)} \\ & \text{corresponding to birth, coalescence and emigration} \\ & \text{(as long as there are more than one particle) of the ith colour individuals.} \end{aligned} \tag{3.433}$$

Next we introduce the appropriate *inter-type coalescence* operators (for versions with formulas for the r.h.s., see (3.438) and sequel below) which describe the changes in the limiting frequency measure U based on changes in the occupation numbers which correspond to actions of the coloured particles. There are essentially two types of intertype-transitions, (1) changes which occur at a site with rates depending on the state at this site and (2) the creation of a new site at rates depending on the state at the founder site leading to the migration operators.

The two groups of matrices corresponding to the coalescence, respectively migration operator, are the following matrices

$$Q^A = (Q^A_{\underline{i},\underline{j}}), \quad \underline{i}, \underline{j} \in \mathbb{N}_0^C \text{ with } A \subseteq C \times C \text{ or } A \subseteq \{a \to b | a, b \in C\} \tag{3.434}$$

running through the list of names of the various transitions are, beginning with the *coalescence* operators:

$$\begin{aligned} Q^{WR} \quad & = \text{coalescence of white and red at same site} \\ & \quad \text{yielding a white and green pair at this site,} \\ Q^{WP} \quad & = \text{coalescence of white and purple at same site} \\ & \quad \text{yielding a white and blue pair at this site,} \\ Q^{PR} \quad & = \text{coalescence of red and purple at same site} \\ & \quad \text{yielding a red at this site,} \\ Q^{RG} \quad & = \text{coalescence of green and red at same site} \\ & \quad \text{yielding a red at this site} \\ & = \text{coalescence of purple and blue at same site} \\ & \quad \text{yielding a purple at this site,} \\ Q^{WG} \quad & = Q^{WB} = 0. \end{aligned} \tag{3.435, 3.436}$$

and the *migration* operators, which are effectively creation operators for new occupied sites, which in particular are then sites occupied by initially only one particle. For sites occupied by *only* one colour we can talk of green (G), blue (B), purple (P), red (R), white (W) sites without ambiguity. However for multicolours, a WR site denotes a site having at least one white or one red particle, etc.

$$Q^{W \to W} \quad = \text{creation of white site at an empty site in } \{1, \ldots, N\}, \tag{3.437}$$

$$
\begin{aligned}
Q^{W\to R} \quad & = \text{creation of red site at randomly chosen occupied site in } \{1,\dots,N\},\\
Q^{R\to R} \quad & = \text{creation of red at empty site in } \{1,\dots,N\},\\
Q^{R\to P,B} \quad & = \text{creation of purple-blue pair}\\
& \quad \text{with purple at randomly chosen occupied site in } \{1,\dots,N\}\\
& \quad \text{and blue at the first empty site in the first copy of } \mathbb{N},\\
Q^{P\to P} \quad & = \text{creation of purple site at randomly chosen site in } \{1,\dots,N\},\\
Q^{G\to G} \quad & = \text{creation of green site at first empty site in the first copy of } \mathbb{N},\\
Q^{B\to B} \quad & = \text{creation of blue site at first empty site in the first copy of } \mathbb{N}.
\end{aligned}
$$

The verbal description on the r.h.s above corresponds to the following expressions which we spell out in two examples:

$$Q^{W\to W}_{(i_W,i_R,i_G,i_B),(1,0,0,0)} = c\cdot 1_{i_W+i_R\neq 1}\, i_W, \tag{3.438}$$

$$Q^{W\to R}_{(i_W,i_R,i_G,i_B),(i'_W,i'_R+1,i'_G,i'_B)} = c\cdot 1_{i_W+i_R\neq 1}\, i_W U_{WRGB}(t,i'_W,i'_R,i'_G,i'_B), \quad \text{etc..} \tag{3.439}$$

We now develop in the detail the system of equations for $U_{WRGB}(t)$ since the former provides the necessary upper and lower bounds.

The pair (u,U) arises as the limit as $N\to\infty$ of the particle system (u^N,U^N) in times

$$T_N + t \text{ where } T_N := \frac{\log N}{\alpha}. \tag{3.440}$$

The proofs are based on first taking the limit as $N\to\infty$ of the $(u^N_{(WRGB)},U^N_{(WRGB)})$ system and then identifying the order of the terms corresponding to the different colours in the limit as $t\to-\infty$ based on the structure of the particle system.

Now we set for the rescaled processes (misusing earlier -notation)

$$
\begin{aligned}
&\widetilde{u}^N_{WRGB}(t) = N^{-1}u^N_{WRGB}(T_N+t),\\
&\widetilde{U}^N_{WRGB}(t,i_W,i_R,i_G,i_B) = U^N_{WRGB}(T_N+t),
\end{aligned} \tag{3.441}
$$

$$
\begin{aligned}
&\widetilde{u}^N_{WRP}(t) = N^{-1}u^N_{WRP}(T_N+t),\\
&\widetilde{U}^N_{WRP}(t,i_W,i_R,i_P) = U^N_{WRP}(T_N+t,i_W,i_R,i_G,i_B).
\end{aligned} \tag{3.442}
$$

Then let $N\to\infty$ and we again obtain, the existence of the limit of the rescaled system (u^N,U^N), the existence of the coloured versions. The limiting objects are denoted:

$$(u_{WRGB}(t),U_{WRGB}(t,\cdot,\cdot,\cdot)), \quad (u_{WRP}(t),U_{WRP}(t,\cdot,\cdot,\cdot)), \tag{3.443}$$

which will satisfy a system of equations which we give below in (3.454), and (3.455) respectively in (3.460). We omit the details of the convergence proof here, which are straightforward modifications of the argument given in the proof of Proposition 3.8.

Remark 36 *In order to give an intuitive picture, even after taking the limit $N\to\infty$, we shall still refer to this nonlinear system in terms of coloured particles, but of course the quantities in question are now continuous quantities that arise in the limit as $N\to\infty$ of the normalized quantities corresponding the the well defined particle system given by the above pure jump process.*

Note first the following facts about the limiting evolution which are important for our purposes and which follow from the corresponding finite-N properties:

- since the green particles compensate for the loss of any red particles due to coalescence, the number of red plus green particles at a given site has an evolution which is independent of the white particles (but not the distribution of the relative proportions of red and green),
- the red plus green plus blue particle system has the same dynamics as the black particle system and since red and black founding particles are created at the same time both systems have the same distribution,
- the particle system comprised of white, red and purple particles has the same distribution as the dual particle system.

Recalling that the WBL-system is a CMJ-system and the RGB process can be identified with the set of black particles, it follows that $(K^N_{(W)}(t) + K^N_{(RGB)}(t))_{t\geq 0}$ is less than a CMJ process with Malthusian parameter α but the difference is of order $o(e^{\alpha t})$ and we have for $t \in \mathbb{R}$, that the intensity of the occupied sites in the various coloured systems satisfies:

$$\begin{aligned}(3.444)\qquad & u_{WR}(t) \leq u_{WRP}(t) \leq u_{WRB}(t) \leq u_{\mathrm{WRGB}} \\ & \leq u_W(t) + u_{RGB}(t) = \lim_{N\to\infty}\left[\frac{1}{N}(u^N_W(T_N+t) + u^N_{RGB}(T_N+t))\right] \leq We^{\alpha t}.\end{aligned}$$

The intensity of individuals in the various coloured systems satisfy that

$$\begin{aligned}(3.445)\qquad & \sum_{i_W,i_R,i_G,i_B} (i_W+i_R) u_{WRGB}(t) U_{WRGB}(t,i_W,i_R,i_G,i_B) \\ & \leq \sum_{i_W,i_R,i_P} (i_W+i_R+i_P) u_{WRP}(t) U_{WRP}(t,i_W,i_R,i_P) \\ & \leq \sum_{i_W,i_R,i_G,i_B} (i_W+i_R+i_B) u_{WRGB}(t) U_{WRBG}(t,i_W,i_R,i_G,i_B).\end{aligned}$$

As a result of these inequalities we note that the $WRGB$ system provides *lower* and *upper* bounds for the "number" of occupied sites and "total number of particles" for the WRP system which corresponds to the dual particle system of interest. We note that the difference between the upper and lower bounds for the total number of particles (which corresponds to the number of blue particles) is of order $O(e^{3\alpha t})$. Since each occupied site must contain at least one particle this provides bounds for the number of occupied sites with the same order of accuracy. For this reason we focus on the WRGB system and then read off the required estimates for the WRP system up to this order of a accuracy.

We now write down the equations for the WRGB-system, i.e. for the corresponding pair (u,U). We need the following abbreviations where all sums are over i_W, i_R, i_G and i_B:

$$(3.446)\qquad u_{(WR)}(t) = u_{(WRGB)}(t)\cdot\left(\sum 1_{i_W+i_R\geq 1} U(t;i_W,i_R,i_G,i_B)\right),$$

$$(3.447)\qquad u_{(WRB)}(t) = u_{(WRGB)}(t)\cdot\left(\sum 1_{i_W+i_R+i_B\geq 1} U(t;i_W,i_R,i_G,i_B)\right),$$

$$(3.448)\qquad \alpha_W(t) = c\sum i_W \cdot 1_{i_W+i_R+i_G\geq 2}\, U_{(WRGB)}(t;i_W,i_R,i_G,i_B),$$

$$(3.449)\qquad \alpha_R(t) = c\sum i_R \cdot 1_{i_R+i_W+i_G\geq 2}\, U_{(WRGB)}(t;i_W,i_R,i_G,i_B),$$

$$(3.450)\qquad \alpha_G(t) = c\sum i_G \cdot 1_{i_G\geq 2}\, U_{(WRGB)}(t;i_W,i_R,i_G,i_B),$$

(3.451) $\alpha_B(t) = c \sum i_B \cdot 1_{i_B \geq 2} U_{(WRGB)}(t; i_W, i_R, i_G, i_B),$

(3.452) $\gamma_W(t) = c 1_{i_W + i_R + i_G = 1} 1_{i_W = 1} U_{(WRGB)}(t; 1, 0, 0, 0),$ similarly for $\gamma_R(t), \gamma_G(t), \gamma_B(t)$.

Then we can write down the equation for $(u_{(C)}, U_{(C)})$ as coloured version of the (u, U)-equation. Again as in the latter case we work in the very same Banach space $(\mathbb{R} \times L_1(\mathbb{N}, \nu), \| \cdot \|)$ without mentioning this explicitly below.

Consider the WRGB-system described by the

(3.453) $(u_{(WRGB)}(t), U_{(WRGB)}(t, \cdot)), t \in \mathbb{R})$

satisfying the following nonlinear equation:

$$\begin{aligned} \frac{\partial u_{(\mathrm{WRGB})}(t)}{\partial t} &= (\alpha_W \to \alpha_W(t) + \alpha_R \alpha_R(t)) u_{(\mathrm{WRGB})}(t)(1 - u_{(\mathrm{WR})}(t)) \\ &\quad + \alpha_G \to \alpha_G(t) u_{(\mathrm{WRGB})}(t) + \alpha_B \to \alpha_B(t) u_{(\mathrm{WRGB})}(t) \\ &\quad - (\gamma_W \to \gamma_W(t) + \gamma_R \to \gamma_R(t)) u_{(\mathrm{WR})}(t) u_{(\mathrm{WRGB})}(t). \end{aligned} \tag{3.454}$$

$$\frac{\partial U_{(\mathrm{WRGB})}(t)}{\partial t} = U_{(\mathrm{WRGB})}(t) Q^{(\mathrm{WRGB})}(t), \tag{3.455}$$

where

$$\begin{aligned} Q^{(\mathrm{WRGB})}(t) = \quad & \sum_{i=W,R,G,B} Q^{0,i} + Q^{(\mathrm{WR})} + Q^{RG} \\ & + (1 - u_{(\mathrm{WR})}(t))[Q^{W \to W} + Q^{R \to R}] + u_{(\mathrm{WR})}(t)[Q^{W \to R} + Q^{R \to B}], \\ & - \alpha(t) I, \end{aligned} \tag{3.456}$$

with initial condition at time $t = -\infty$ given by

$$\lim_{t \to -\infty} u_{(\mathrm{WRGB})}(t) = 0, \tag{3.457}$$

$$\begin{aligned} \lim_{t \to -\infty} U_{\mathrm{WRGB}}(t, i_W, i_R, i_G, i_B) \quad & = \mathcal{U}_\infty(i_W) \quad \text{if } i_R = i_G = i_B = 0 \\ & = 0, \quad \text{otherwise}, \end{aligned} \tag{3.458}$$

where $\mathcal{U}_\infty$ is the stable size distribution of the CMJ process.

If we write for a better understanding of (3.455) this equation out pointwise, we obtain (we suppress on the r.h.s. the subscript WRGB in U):

(3.459)

$$\begin{aligned} & \frac{\partial U_{(\mathrm{WRGB})}}{\partial t}(t; i_W, i_R, i_G, i_B) = \\ & + \frac{d}{2} i_W (i_R + 1) U(t; i_W, i_R + 1, i_G - 1, i_B) - \frac{d}{2} i_W i_R U(t; i_W, i_R, i_G - 1, i_B) \\ & + \frac{d}{2}(i_W + i_R)(i_G + 1) U(t; i_W, i_R, i_G + 1, i_B) \\ & - \frac{d}{2}(i_W + i_R) i_G 1_{i_R \neq 1} U(t; i_W, i_R + 1, i_G - 1, i_B) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_W(t)(1-u_{(WR)}(t))1_{(i_{W=1,i_R=0,i_G=0,i_B=0})}\\
&+\alpha_R(t)(1-u_{(WR)}(t))1_{(i_{W=0,i_R=1,i_G=0,i_B=0})}\\
&+(\alpha_W(t)+\gamma_W(t))u_{(WR)}(t)(U(t;i_W,i_R+1,i_G-1,i_B)-U(t;_W,i_R+1,i_G-1,i_B)\\
&+(\alpha_G(t)+\sum_{i+j\geq 1}U(t;i_W,i_R+1,i_G-1,i_B))1_{(0,0,1,0)}\\
&+(\alpha_B(t)+\sum_j jU(t;i_W,i_R+1,i_G-1,i_B)u_{(WR)}(t))1_{(0,0,0,1)}\\
&-(\alpha(t)(1-u_{WR}(t))-\gamma(t)u_{WR}(t))\cdot U_{(WRGB)}(t;i_W,i_R+1,i_G-1,i_B).
\end{aligned}
$$

The process $(u_{(WRP)},U_{(WRP)})$ satisfies for $u_{(WRP)}=u$ our original equation, and the quantity $U_{(WRP)}(t)$ satisfies a similar set of equations as above with B replaced by P but in this case we must add the terms corresponding to the migration of a purple particle to a site occupied by white, red or purple. Also when a purple coalesces with red at the same site (which occurs with rate d) it produces a red-green pair and when it coalesces with a white at the same site it produces a white-green pair. The equation for $U_{(WRP)}$ has therefore the form

$$
\frac{\partial U_{(WRP)}(t)}{\partial t}=U_{(WRP)}(t)Q^{WRP}(t), \tag{3.460}
$$

where

$$
\begin{aligned}
Q^{WRP}(t)= \quad & \sum_{i=W,R,P}Q^{0,i}+Q^{WR}+Q^{WP}+Q^{PR}\\
&+c(1-u(t))[Q^{W\to W}+W^{R\to R}]+cu(t)[Q^{W\to R}+Q^{R\to P}]\\
&+cQ^{P\to P}-\alpha(t)I.
\end{aligned} \tag{3.461}
$$

This now completes the set of limiting ($\mathbb{N}\to\infty$) equations for the coloured particle system at time T_N+t.

For the further analysis it is crucial to observe the following two facts:

- for calculating the asymptotics of *first* moments of the McKean-Vlasov system it suffices to isolate dual process effects which in the limit $N\to\infty$ are of the order $e^{\alpha t}$ as $t\to-\infty$ and ignore error terms of order $e^{2\alpha t}$ and higher,
- for calculating the asymptotics of *second* moments of the McKean-Vlasov system it suffices to isolate effects which are in the limit $N\to\infty$ of the order $e^{2\alpha t}$ as $t\to-\infty$ and ignore error terms of order $e^{3\alpha t}$ and higher.

It is easily verified that if we *condition* on the growth constant W and on W_*, we can verify that as $t\to-\infty$, we have the following estimates on the "number of sites" occupied by certain colour combinations:

$$
\sum_{i_W,i_R,i_G,i_B}1_{i_R+i_G+i_B\geq 1}U_{(WRGB)}(t;i_W,i_R,i_G,i_B)=O(e^{\alpha t}), \tag{3.462}
$$

$$
\sum_{i_W,i_R,i_G,i_B}1_{i_B\geq 1}U_{(WRGB)}(t;i_W,i_R,i_G,i_B)=O(e^{2\alpha t}) \tag{3.463}
$$

and therefore

$$
u_{RGB}(t)=O(e^{2\alpha t})\text{ and }u_B(t)=O(e^{3\alpha t}). \tag{3.464}
$$

We conclude showing that as $t \to -\infty$ this difference between the red plus white particle system (with WRGB dynamics) and the true dual (WRP) is of order $o(e^{2\alpha t})$ once we have taken the limit $N \to \infty$. This difference is the difference between the system of *purple* particles and the system of *blue* particles.

Again we condition on W and W_*. We first note that the intensity (after taking $N \to \infty$) of red particles is $O(e^{2\alpha t})$ and therefore the rate of production for collisions of red particles with other occupied sites by white, red or purple is $O(e^{3\alpha t})$. Hence we have as $t \to -\infty$

$$u_P(t) = O(e^{3\alpha t}), \quad u(t) - u_{(WRGB)}(t) = O(e^{3\alpha t}). \tag{3.465}$$

The sites occupied by red, green or blue particles correspond to black sites and therefore bound the number of occupied sites eventually lost (compared to CMJ) due to collisions. What is the meaning of the three components? The population of green particles represent asymptotically as $t \to -\infty$ the total number of particles lost (in the dual particle system compared with the collision-free CMJ-system) due to collisions up to order $o(e^{2\alpha t})$.

Remark 37 *It is important to remember that we have seen above in (3.444), (3.445) that we can obtain upper and lower bounds for the exact dual using* white plus red *for the lower bound and* white plus red plus blue *for the upper bound and that by (3.465) we know that the difference of lower and upper bound is $O(e^{3\alpha t}) = o(e^{2\alpha t})$. Hence suffices for the purpose of first and second moment calculations for the random McKean-Vlasov entrance law to work with the WRGB-system on the dual side.*

Step 3:
Proof of Proposition 3.9

(a) We define the coupling using the multicolour system where white and black gives the CMJ-process and WRP the dual particle system. The evolution rules define the processes in a standard way for all N on one probability space using Poisson stream for all potential sites, colours and transitions.

Recall that the total number of sites occupied by black plus white particles can be identified with a CMJ process K_t and our processes for each N can be coupled based on a single realization of this CMJ process using the multicolour system. Let on this common probability space

$$W := \lim_{t\to\infty} e^{-\alpha t} K_t. \tag{3.466}$$

Then the existence of the limit in (3.466) and therefore assertion (3.375) follows immediately from the Crump-Mode-Jagers theory.

In order to obtain the assertion (3.376) we shall estimate below the *mean* and *variance* of

$$e^{-\alpha t} N^{-1} u^N(T_N + t) \qquad (\text{recall } T_N = \alpha^{-1} \log N) \tag{3.467}$$

by bounding the mean and variance of the normalized number of black particles (equivalently, red plus green plus blue particles) and finally put this together to get the claim of the proposition.

Part 1: Bounds on the mean.

Define here

$$W_t = e^{-\alpha t} K_t, \quad W_{N,t} = W_{T_N+t}, \quad W = W_\infty. \tag{3.468}$$

Recall that by CMJ theory ([N]), and finiteness of $E[W^2]$,

$$\bar{w} = \sup_t E[W_t] < \infty, \quad \overline{w^2} = \sup_t E[W_t^2] < \infty. \tag{3.469}$$

We condition now on the path $(K_t)_{t\geq 0}$ (and then in particular W is given) and give a conditional upper bound for the number of black particles produced by time T_N+t. This bound is given by considering the following upper bound for the rate of collisions of white particles at time s which is

$$(3.470) \quad (\alpha_N(s)+\gamma_N(s))K_s^N \frac{K_s^N}{N},$$

where $\alpha_N(\cdot)$ and $\gamma_N(\cdot)$ are defined as in (3.239). Recall that that $\alpha_N(s)\to\alpha(s)$, $\gamma_N(s)\to\gamma(s)$ and $(\alpha(s)+\gamma(s))K_s \leq (\alpha+\gamma)W_s e^{\alpha s}$ as $N\to\infty$. We get a stochastic upper bound of the quantity in (3.470) by taking the random process

$$(3.471) \quad (\alpha+\gamma)W_s e^{\alpha s}\frac{K_s^N}{N}.$$

Now assume that we realize a Poisson point process on $[0,\infty)$ with intensity measure given by the density given in (3.470). Now we use this Poisson point process to generate the founders of the black families. Then we let a black cloud grow descending from one black ancestor born at time s up to time (T_N+t-s) and this evolution is independent of the Poisson point process. This black cloud born at time s grows till time T_N+t as

$$(3.472) \quad \widetilde{W}_{N,s}(t)e^{\alpha(t-s)}$$

and is given by an independent copy of the CMJ process starting with one particle at time s. With this object we obtain a stochastic upper bound on the number of black particles in the original dual process.

Since at every time s of birth of a black particle we get a cloud independent of all other clouds and also independent of $(K_s)_{s\geq 0}$, we get an upper bound for the expected number of black particles *given a realization of* $(K_s)_{s\geq 0}$:

$$(3.473) \quad \int_0^{T_N+t} E[\widetilde{W}_{N,t-s}]e^{\alpha(T_N+t-s)}(\alpha_N(s)+\gamma_N(s))K_s^N\frac{K_s^N}{N}ds.$$

Furthermore with the Crump-Mode-Jagers theory applied to $(K_s,\mathcal{U}(s))$ we get setting $D=E[\widetilde{W}]\frac{\alpha+\gamma}{\alpha}\in(0,\infty)$ that the quantity in (3.473) can be bounded above asymptotically by (recall (3.468) for $W_{N,s}$):

$$(3.474) \quad \frac{1}{N}\int_0^{T_N+t} e^{\alpha(T_N+t-s)}\alpha D W_s^2 e^{2\alpha s}ds \quad \text{as } N\to\infty.$$

The *mean* of the expression in (3.474) is bounded by

$$(3.475) \quad \overline{w^2}\alpha D\frac{1}{N}\int_0^{T_N+t} e^{\alpha(T_N+t-s)}e^{2\alpha s}ds.$$

This quantity in turn is equal to

$$(3.476) \quad N\overline{w^2}e^{2\alpha t}(1-e^{-\alpha(T_N+t)}) = ND\overline{w^2}e^{2\alpha t}(1-e^{-\alpha t}\frac{1}{N}).$$

Therefore we obtain the upper and lower bound:

$$(3.477) \quad \bar{w}e^{\alpha t}\geq E[\frac{1}{N}u^N(T_N+t)]\geq E[W_{N,t}]e^{\alpha t}-\overline{w^2}e^{2\alpha t}(1-O(\frac{1}{N})),$$

where the last expression is an upper bound on the expected number of black particles.

Therefore we get the final bound for the mean of the normalized number of sites:

$$0 \geq E[e^{-\alpha t} N^{-1} u^N(T_N + t) - W_{N,t}] \geq c_N \cdot e^{\alpha t}(1 - O(\frac{1}{N})) \tag{3.478}$$

where $\sup_N(|c_N|) < \infty$.

Part 2: Analysis of the variance
To complete the result we now have to estimate

$$Var[e^{-\alpha t} N^{-1} u^N(T_N + t) - W_{N,t}]. \tag{3.479}$$

We will verify as a first step that the variance of the *normalized* number of black particles converges to 0 as $N \to \infty$ and then we will return to the dual particle system. Recall that in our calculations we condition on $(K_t)_{t \geq 0}$.

The first step is now to condition again on a realisation of a process which is a stochastic upper bound on the number of white particles (compare part 1). This bound is given by a CMJ-process. Therefore we observe first that given this number the evolution of the black clouds once they are founded are all independent. Therefore we obtain an *upper bound* on the normalized variance of the black particles if we use a path of the CMJ-process, which dominates the white population.

Note that therefore the *birth of new black clouds* of particles as $N \to \infty$ (which arises upon collision) can by (3.474) be bounded by a time inhomogeneous Poisson process with intensity

$$\left(\alpha \frac{W_{N,t}^2 e^{2\alpha t}}{N} \right) ds, \tag{3.480}$$

where we condition on a realisation of $W(t), W_{N,t} = W(T_N + t)$, or alternatively on the path $(K_s)_{s \geq 0}$.

Define L_s^r as (for given r in the variable s) the Laplace transform of $e^{-\alpha(r-s)} K_{r-s}$ (starting with one particle).

The Laplace transform L of the total number of black particles at time r then is given by

$$L^r(\lambda) = \exp\left(- \int_0^r D \frac{\alpha (K_s^N)^2}{N} (1 - L_s^r(\lambda e^{\alpha(r-s)})) ds \right). \tag{3.481}$$

Note that by construction $L_s^r = L_{r-s}$. Furthermore we have that:

$$\sup_{s \geq 0} Var[e^{-\alpha s} K_s] = \sup_{s \geq 0} L_s''(0) < \infty. \tag{3.482}$$

Now we apply this to $r = T_N + t$ and conclude that conditioned on W (this is indicated by $\sim$ on E, Var, etc.):

$$\widetilde{V}ar[e^{-\alpha t} N^{-1} \cdot (\#\text{ of black particles at time } T_N + t)] \tag{3.483}$$

is bounded by (Const means here a function of W only)

$$\frac{e^{-2\alpha t}}{N^2} \int_0^{T_N + t} D W^2 \alpha e^{2\alpha(T_N + t - s)} \frac{e^{2\alpha s}}{N} L_s''(0) ds \leq Const \frac{N^2 (\log N)}{N^3}. \tag{3.484}$$

The r.h.s. is independent of t. Hence *uniformly* in $t \in \mathbb{R}$, as $N \to \infty$:

$$\widetilde{V}ar[e^{-\alpha t} \frac{1}{N} (\#\text{ black particles at time } T_N + t)] = o(1). \tag{3.485}$$

Hence *conditioned on* W the number of black and white particles normalized by $Ne^{\alpha t}$ is *deterministic* in the limit $N \to \infty$. Hence by (3.485) the density of white particles alone and the normalized difference between the number of black and white particles is deterministic in the $N \to \infty$ limit.

Part 3 Conclusion of argument

Since the dual lies between the set of CMJ particles and the set of white particles, conditioned on the variable W, in the limit $N^{-1}e^{-\alpha t}u^N(T_N + t)$ lies between *two deterministic curves*, both converging to the same constant as $t \to -\infty$, namely W. Using (3.477) and (3.478) we conclude inequality (3.376).

The assertion (3.377) is proved as follows. Since the first component was just treated, we focus on the second, i.e. U^N. Note that the total variation distance between $U^N(T_N+t)$ and $\mathcal{U}(T_N+t)$ is bounded by the normalized number of black particles. Then combining (3.473) with (3.476) we have

$$\lim_{t\to-\infty} \|U^N(T_N + t) - \mathcal{U}(T_N + t)\| = 0. \tag{3.486}$$

(b) is an immediate consequence of the analysis above.

Step 4:

Proof of Proposition 3.10

We proceed in six parts. Four parts prove the various claims and two parts are needed to prove some key lemmata at the end of the argument.

Part 1 - Proof of (3.379)

To verify (3.379) we use that the $u(t), \alpha(t), \gamma(t)$ arise as the limit $N \to \infty$ of $\frac{u^N(t)}{N}, \alpha^N(t), \gamma^N(t)$ for which we can use the representation by the multicolour particle system. For each N the inequality

$$(\alpha_N(t) + \gamma_N(t))u^N(t) \le (\gamma + \alpha)NW_{N,t}e^{\alpha t} \tag{3.487}$$

follows since the left side counts the number of particles in a subset of the set of particles on the right side (the difference is induced by the green particles in the multicolour construction). Furthermore we know that $W_{N,t} \to W$, a.s.. Hence the inequalities in (3.487) are therefore inherited in the limit as $N \to \infty$ and give

$$(\alpha(t) + \gamma(t))u(t) \le (\alpha + \gamma)We^{\alpha t}. \tag{3.488}$$

Part 2 - Proof of (3.381), (3.386)

To obtain (3.381), (3.386) we use the coloured particle system introduced in Step 1 of this subsection. We show first that κ and κ^* are strictly positive and then later on we identify these numbers by a more detailed analysis.

Recall that the total number of white, red, green and blue particles at time $T_N + t$ equals the total number, $W_{N,t}Ne^{\alpha t}$ in the CMJ process and the collection of white, red and purple particles is a version of the actual dual particle system η^N. We have shown that if we consider these particle systems at time $T_N + t$ and let $N \to \infty$ we obtain a limit dynamic. The analoguous statement holds for the limiting dynamic as $N \to \infty$, now with t as the time variable for the multicolour system. The techniques are the same as used in Subsubsection 3.2.8 and we do not write out here the details again. We are now interested in the expansion as $t \to -\infty$ in this multicolour limit dynamics.

In contrast to the proof of Proposition 3.9 the argument here now involves the green and blue particles.

The purpose of the green particles is to keep track of the particles *lost due to collisions* of white with red and purple particles (lost by coalescence). The purpose of the blue particles is to obtain

with the WRB-system an upper bound for the total number of particles in the WRP-system by suppressing the loss of particles that could occur if a red or purple particle migrates to an occupied site. We observe that therefore that since the number of blue particles is $O(e^{3\alpha t})$ (recall (3.464)), then in order to identfiy the $e^{2\alpha t}$ term in the total number of dual particles as $t \to -\infty$, it is sufficient to control the asymptotics as $t \to -\infty$ of the green particles. Therefore we now have to study the green population, for which we first need more information about the red particles.

The rate at which white particles migrate and collide with occupied sites thus creating a red particle is given and estimated as follows (recall (3.488)):

$$(3.489)\quad (\alpha_W(t)+\gamma_W(t))u_W^2(t) \le (\alpha_W(t)+\gamma_W(t))u^2(t) \le W^2(\alpha+\gamma)e^{2\alpha t}.$$

At the particle level, once a red particle is created it begins to develop a growing cloud of red sites which grows with exponential rate $\le \alpha$. Using the latter and (3.489) we obtain

$$(3.490)\quad \sum_{i_W,i_R,i_P} 1_{i_R\ge 1} U_{WRP}(t,i_W,i_R,i_P) = O(e^{2\alpha t}).$$

Return now to the production of the green particles. We will show first that the number of green particles and sites with green particles is of order $e^{2\alpha t}$ as $t \to -\infty$.

We can assume that when a red particle is created on an occupied site the number of white particles is given by the stable age distribution by (3.458). (We will verify below in (3.559) that in fact the error in (3.458) is $O(e^{\alpha t})$.) We now want to study the production of sites with green particles. We introduce the concept of special sites for this purpose.

We briefly return to the finite N-system. We call occupied sites (by a white particle) in the coloured particle system

(3.491) *"special sites" at time s*

if s lies between the (random) time when a first red particle arrives at this site until it contains no red, purple or green particles. This random time is a.s. finite.

We transfer this concept to the $N \to \infty$ limit at time $T_N + t$. This means that we incorporate the additional mark in the measure-valued description.

During the lifetime of a special site a special site can produce red, purple and green migrants. Special sites are created at rate (recall for the first equality sign that α, γ are the limits of $\alpha(t), \gamma(t)$ as $t \to -\infty$, i.e. the ones characterizing the collision-free regime and then use (3.464) to get $o(1)O(e^{2\alpha t})$ as bound for the difference of both sides):

$$(3.492)\quad (\alpha_W(t)+\gamma_W(t))u_W^2(t) = (\alpha+\gamma)u^2(t) + o(e^{2\alpha t}) = W^2(\alpha+\gamma)e^{2\alpha t} + o(e^{2\alpha t})$$

and therefore by integration from $-\infty$ to t the number of special sites created up to time t is bounded below by

$$(3.493)\quad W^2 \frac{\alpha+\gamma}{2\alpha} e^{2\alpha t} + o(e^{2\alpha t}).$$

Then green particles at a special site are produced at rate

$$(3.494)\quad d \sum_{i_W,i_R,i_B} i_W(i_R+i_P)\, U_{WRP}(t;i_W,i_R,i_P) \ge d \sum_{i_W,i_R,i_P} 1_{i_W(i_R+i_P)\ge 1} U_{WRP}(t;i_W,i_R,i_P).$$

Since $i_W(i_R+i_P) \ge 1$ automatically at a special site we use (3.492) and (3.493), to get that the expected number of green particles at time t is bounded below by

$$(3.495)\quad d \cdot W^2 \frac{\alpha+\gamma}{2\alpha} e^{2\alpha t} + o(e^{2\alpha t}).$$

Provided the limit exists (see below), this implies that (recall the green particles describe the loss of the WRP-system compared to the W-BL-system)

$$\lim_{t\to-\infty} e^{-2\alpha t}[(\alpha+\gamma)We^{\alpha t} - (\alpha(t)+\gamma(t))u(t)] = \kappa^* > 0. \tag{3.496}$$

In order to now get also information on κ, we need information on sites rather than on particle numbers. In order to get that the number of sites with only green particles is of order $e^{2\alpha t}$ as $t \to -\infty$, we argue as follows. Since there is a positive probability that a green particles migrates before coalescing with a red or white, this implies (if the limit exists, see below) that

$$\lim_{t\to-\infty} e^{-2\alpha t}[(We^{\alpha t} - u(t)] = \kappa > 0. \tag{3.497}$$

Hence we now know (provided that the limits taken exist) that:

$$\kappa, \kappa^* > 0. \tag{3.498}$$

Next, in order to identify the constant κ, κ^* we need to obtain the 2nd order asymptotics. It remains therefore to determine the actual value of κ, κ^* using this information.

We first consider the production of blue particles. Since there are $O(e^{2\alpha t})$ special sites and a blue particle is created only when a migrant comes from a special site and hits the set of size $u(t)$ of occupied sites, it follows that the number of blue particles produced is of order $O(e^{3\alpha t})$ and hence indeed the *blue* particles and therefore also the *purple* particles play *no role* determining the $e^{2\alpha t}$-term. In particular the production of green particles by the loss of purple particles is of order $O(e^{3\alpha t})$ and can be omitted.

We first obtain an expression for κ^*. Since the green particles represent the particles lost in the WRP-system due to coalescence of red or purple with white particles, this is obtained by considering the growth of the green particles in more detail. We note that the loss of purple particles is of smaller order than the loss of the red particles as mentioned above. As a result, in order to to identify the constant κ or κ^* in the expressions for $u(t)$ (number of sites) or $(\alpha(t)+\gamma(t))u(t)$ (number of particles) we can work with the WRGB-system instead of the WRP-system. Hence in carrying out the analysis using the WRGB-system we obtain a lower bound for the number of particles lost but as shown above the resulting error is $O(e^{3\alpha t})$.

Once a new green particle is produced by a white-red coalescence at a special site we are interested in the number of green particles that migrate before the end of the life time of the special site. As noted above we can assume that when the red arrives the number of white particles is given by the stable size distribution. The number of green particles at the special site and the process of producing green migrants can then be obtained by the analysis of a *modified birth and death* process where we now have two types, namely, red and green with birth and death rules inherited from the WRGB dynamics. In terms of the limit process this involves the forward Kolmogorov equations for this modified birth and death process which serves as a source of migrants for the green particle system which then evolves by the CMJ dynamics.

To summarize, in order to obtain the distribution of green mass up to an error term of order $O(e^{3\alpha t})$ we consider

- the WRGB-system instead of the WRP-system, ignore blue particles
- the production of special sites by red-white collision,
- the production of green particles and green migrants at a special site,
- a CMJ process with immigration, with immigration source given by the green migrants from special sites.

Recall from the derivation of a lower bound on κ above that in the $WRGB$-system red always migrates to a new (unoccupied) site or otherwise a purple-blue pair is created so that the number of *white plus red and blue* particles gives an *upper bound* to the original interacting dual particle system in the limit $N \to \infty$. The number of the *green particles* produced only by white-red coalescence (omitting purple-white coalescence) gives up to an error of order $O(e^{3\alpha t})$ a *lower bound* to $W(\alpha+\gamma)e^{\alpha t} - (\alpha(t)+\gamma(t))u(t)$.

We make the following definitions in order to turn bounds into precise asymptotics. We say below "expected", to distinguish from the usual expected value, when we calculate quantities of the form $\sum i_G U(t, i_W, i_R, i_G, i_B, i_P)$, where the sum is overall $i_A, A \in C$. Let

$$g_1(t,s) \tag{3.499}$$

be the "expected" number of green particles at time t at a special site created at time s. Note that a special site has a finite lifetime (with finite expected value) since due to coalescence it will revert to a single white particle at some finite random time after its creation. Therefore the function $g_1(t,s) = g_1(t-s)$ is bounded.

Let

$$g_2(s,r)dr \tag{3.500}$$

be the rate of production of green migrants at time r at a special site created at time, i.e. the c times the "expected" number of green particles at special sites. This function is also bounded as above.

Let

$$g_3(r,t) \leq \text{const} \cdot \mathrm{e}^{\alpha(\mathrm{t}-\mathrm{r})} \tag{3.501}$$

be the "expected" number of green *sites* produced at time t from a founder at time r and finally

$$g_3^*(r,t) \leq \text{const} \cdot \mathrm{e}^{\alpha(\mathrm{t}-\mathrm{r})} \tag{3.502}$$

is the "expected" number of green *particles* produced at time t from a founder at time r. These four functions determine the numbers κ and κ^* uniquely as we shall see next.

Now the creation of the first green particles at a site and then subsequently green sites occurs from a red-white coalescence at a site. At such an event a green particle arises and if there is no green particle yet a new green site is created at this moment. From these founders now a cloud of green particles and sites develops. The evolution of these clouds is independent of the further development of the number of white particles and white sites. Therefore conditioned on W we get that the *"expected" number of green particles at time* t, denoted κ_t^* is asymptotically as $t \to -\infty$

$$W^2 \kappa^* e^{2\alpha t} + O(e^{3\alpha t}), \tag{3.503}$$

where $\kappa^* = \lim\limits_{t \to -\infty} \kappa_t^*$ and

$$\kappa_t^* = e^{-2\alpha t} \int_{-\infty}^t (\alpha+\gamma) e^{2\alpha s} \left[g_1(t,s) + \left(\int_s^t g_2(s,r) g_3^*(r,t) dr \right) \right] ds + O(e^{\alpha t}). \tag{3.504}$$

Since g_1 and g_2 are bounded, the integral is finite. Observe that $g_1(t,s) = \widetilde{g}_1(t-s)$ and $g_2(s,r) = \widetilde{g}_2(r-s), g_3^*(r,t) = \widetilde{g}_3^*(t-r)$ by construction of the dynamic of the green particles, which do not coalesce with white or red particles. Therefore the first term in (3.504) is independent of t. The second part is given by an integral depending again only on $t-s$ and hence the complete term again does not depend on t. This implies the convergence of κ_t^* as $t \to -\infty$.

We now turn to the identification of κ that is we turn from particle numbers to number of sites. As in the identification of κ^* we can show that there exists κ such that

$$u_{WR}(t) = We^{\alpha t} - \kappa W^2 e^{2\alpha t} + O(e^{3\alpha t}). \tag{3.505}$$

We obtain the constant κ by counting *green sites* (i.e. sites which are not also occupied by only red or white) instead of green particles. Conditioning on W we get the *"expected" number of green sites*, at time t is asymptotically as $t \to -\infty$:

$$W^2 \kappa e^{2\alpha t} + O(e^{3\alpha t}), \tag{3.506}$$

where $\kappa = \lim\limits_{t \to -\infty} \kappa_t$ with

$$\kappa_t = e^{-2\alpha t} \int_{-\infty}^{t} (\alpha + \gamma) e^{2\alpha s} \left[\left(\int_s^t g_2(s,r) g_3(r,t) dr \right) \right] ds + O(e^{3\alpha t}). \tag{3.507}$$

The existence of the limit follows as above.

Part 3 - Proof of (3.380), (3.382), (3.389)

Next we return to our original nonlinear equation, which we now relate with the quantities of our multicolour particle system. We will need to collect in (3.509)-(3.514) some facts on this system used in the proof.

We first note the relation between the original dual and the WRP system and the possibility to replace them with the WRGB-system for the asymptotic as $t \to -\infty$, namely as $t \to -\infty$:

$$u(t) = u_{\mathrm{WRP}}(t) = u_{\mathrm{WR}}(t) + O(e^{3\alpha t}), \tag{3.508}$$

$$\begin{aligned} U(t,k) \quad &= \sum_{i_W, i_R, i_P} 1_{i_W + i_R + i_P = k} U_{\mathrm{WRP}}(t, i_W, i_R, i_P) \\ &= \frac{u_{\mathrm{WRGB}}(t)}{u(t)} \sum_{i_W, i_R, i_G, i_B} 1_{i_W + i_R = k} U_{\mathrm{WRGB}}(t; i_W, i_R, i_G, i_B) + o(1). \end{aligned} \tag{3.509}$$

Here the first factor arises since the normalization (by the number of sites) for the WRP system is $u(t)$ and it is $u_{\mathrm{WRGB}}(t)$ for the WRGB system. We also recall that by construction

$$u_{\mathrm{WR}} \leq u = u_{\mathrm{WRP}} \leq u_{\mathrm{WRB}}. \tag{3.510}$$

Recall furthermore that the sites occupied by black particles are in one-to-one correspondence with the sites occupied by the red, green and blue particles and the number of white plus black sites is $W_t e^{\alpha t}$ with $W_t \to W$ as $t \to \infty$. Therefore the difference in the number of occupied sites in the (W-BL)-system and the WRGB-system arises from white particles sharing a site with the coloured particles, but those sites are represented as two sites in the (W-BL)-system. Hence

$$\frac{u_{\mathrm{WRGB}}(t)}{We^{\alpha t}} \quad = \frac{\sum_{i_W, i_R, i_G, i_B} [1_{i_W + i_R + i_G + i_B \geq 1}] U_{\mathrm{WRGB}}(t, i_W, i_R, i_G, i_B)}{\sum_{i_W, i_R, i_G, i_B} [1_{i_W \geq 1} + 1_{i_R + i_G + i_B \geq 1}] U_{\mathrm{WRGB}}(t, i_W, i_R, i_G, i_B)}. \tag{3.511}$$

Furthermore the estimates on the number of green, red and purple particles in the part 2 of our argument for Proposition 3.10 imply as well

$$\frac{u_{\mathrm{WRGB}}(t)}{u_{\mathrm{WRP}}(t)} = 1 + \kappa We^{\alpha t} + O(e^{2\alpha t}) \tag{3.512}$$

and

$$\alpha(t) = \frac{u_{\text{WRGB}}(t)}{u_{\text{WRP}}(t)}(\alpha_W(t) + \alpha_R(t)) + O(e^{2\alpha t}), \tag{3.513}$$

$$\gamma(t) = \frac{u_{\text{WRGB}}(t)}{u_{\text{WRP}}(t)} \sum_{i_W, i_R, i_G, i_B} (1_{i_R + i_W = 1} U_{\text{WRGB}}(t, i_W, i_R, i_G, i_B)) + O(e^{2\alpha t}). \tag{3.514}$$

Using the equations (3.508-3.514) we shall show below that (conditioned on W) we have the following approximation relations of $t \to -\infty$ for u, α, γ and U, which finish the proof of (3.380), (3.382) and (3.386):

$$|u(t) - We^{\alpha t}| = O(e^{2\alpha t}), \tag{3.515}$$

$$|\alpha(t) - \alpha| = O(e^{\alpha t}), \tag{3.516}$$

$$|\gamma(t) - \gamma| = O(e^{\alpha t}), \tag{3.517}$$

$$\sum_{k=1}^{\infty} | \sum_{i_W, i_R, i_G, i_B} 1_{i_R + i_W = k} U_{\text{WRGB}}(t, i_W, i_R, i_G, i_B) - \mathcal{U}(\infty, k))| = O(e^{\alpha t}). \tag{3.518}$$

It therefore remains now to verify (3.515)-(3.518) in the sequel.

The bound (3.515) follows from the fact that the difference between the W-BL-system giving $We^{\alpha t}$ and the WRP-system giving $u(t)$ is bounded by the RGB-system which satifies $u_{RGB}(t) = O(e^{2\alpha t})$.

Turn next to the proof of (3.516). First note that the bound of the WRP-system (for which $\alpha(t)$ is the rate of colonization of new sites by multiple occupied sites) from below and above by the WR-system respectively the WRB-particles from the WRGB-system yields by dividing the inequality by $u_{WRP}(= u)$:

$$\frac{u_{\text{WRGB}}(t)}{u_{\text{WRP}}(t)}(\alpha_W(t) + \alpha_R(t)) \leq \alpha(t) \leq \frac{u_{\text{WRGB}}(t)}{u_{\text{WRP}}(t)}(\alpha_W(t) + \alpha_R(t) + \alpha_B(t)) \tag{3.519}$$

The difference between the first and third expressions in (3.519) is $O(e^{2\alpha t})$ (bounding $\alpha_B(t)$ by $O(e^{2\alpha t})$ and using (3.512)). Therefore we get using the representation of $\alpha(t)$ by the WRP-system (compare (3.513)):

$$\begin{aligned} \alpha(t) \quad = \quad & \sum_{i_W, i_R, i_G, i_B} [1_{i_W > 1}(i_W) + 1_{i_W = 1, i_R > 0}(i_W + i_R) + 1_{i_W = 0, i_R > 1}(i_R)] \\ & \qquad \cdot U_{\text{WRGB}}(t, i_W, i_R, i_G, i_B) \cdot \frac{u_{\text{WRGB}}(t)}{u_{\text{WRP}}(t)} + O(e^{2\alpha t}). \end{aligned} \tag{3.520}$$

We can obtain the expression for α, which is suitable for the comparison with $\alpha(t)$, as follows. This constant α arises from the (W-BL)-system. Furthermore the RGB-particles are in correspondence with the black particles, only sit always on the copy of $\mathbb{N}$ instead of $\mathbb{N}$ or $\{1, \cdots, N\}$. (Note

that we are interested here on particle numbers!) Therefore since $W - BL$ is a CMJ-system in the stable age-type distribution

$$\begin{aligned}\alpha = \quad & u_{WRGB}(t)(We^{\alpha t})^{-1} \\ & (\sum_{i_W,i_R,i_G,i_B} [1_{i_W>1} i_W + 1_{i_W=0} 1_{i_R+i_G+i_B>1}(i_R+i_G+i_B) \\ & \qquad + 1_{i_W=1,i_R+i_G+i_B>0}(i_W+i_R+i_G+i_B) \\ & \qquad + 1_{i_W>1}(i_R+i_G+i_B)] U_{WRGB}(t,i_W,i_R,i_G,i_B)).\end{aligned} \tag{3.521}$$

Therefore we can represent $\alpha(t) = \alpha + \widetilde{\alpha}(t)$ and by combining (3.520) and (3.521) together with the fact that blue particles are $O(e^{3\alpha t})$, we get an expression for $\widetilde{\alpha}(t)$ which together with (3.512) results as $t \to -\infty$ in

$$\alpha(t) = \alpha + O(e^{\alpha t}), \tag{3.522}$$

which proves (3.516). (See Lemma 3.15 below for the characterization of $\widetilde{\alpha}(t)$.)

Similarly we can proceed with the remaining claims (3.517) and (3.518). This completes the proof of (3.380), (3.382) and (3.389) as pointed out below (3.514).

Part 4 - Proof of (3.384) and (3.385)

For this purpose we must investigate the difference between $\alpha(t)$, $U(t)$, $\gamma(t)$ and α, $\mathcal{U}(\infty)$, γ, respectively more accurately than in the bounds above. Namely we need an expression with error terms of order $O(e^{3\alpha t})$ resp. $O(e^{2\alpha t})$ in (3.515) resp. (3.516)-(3.518). For that purpose we use again the multicolour representation. We begin by establishing the order of these differences rather than only upper bounds. This is based on the coloured particle system. From the above discussion we expect that $\widetilde{\alpha}(t) = \alpha(t) - \alpha \sim \widetilde{\alpha}_0 e^{\alpha t}$ for some $\widetilde{\alpha}_0 > 0$ which we have to identify.

Recall that the intensity of green particles at time t corresponds to

$$(\alpha + \gamma)e^{\alpha t} - (\alpha(t) + \gamma(t))u(t) \tag{3.523}$$

and the number of green particles is given W of order $O(e^{2\alpha t})$ so that conditioned on W we have

$$\begin{aligned}&(\alpha+\gamma)We^{\alpha t} - (\alpha(t)+\gamma(t))u(t) \geq \delta e^{2\alpha t}, \quad -\infty < t \leq t_0, \\ &\text{for some positive constant } \delta.\end{aligned} \tag{3.524}$$

We want to sharpen this inequality above to a precise second order expansion. To do this and to thereby identifying $\widetilde{\alpha}_0$, $\widetilde{\gamma}_0$ we now return to the analytical study of the nonlinear system (3.414), (3.415) and prove the following.

Lemma 3.15 *(Expansion of $\alpha(t)$ and $\gamma(t)$)*

(a) Let

$$\widetilde{\alpha}(t) := \alpha(t) - \alpha, \quad \widetilde{\gamma}(t) = \gamma(t) - \gamma. \tag{3.525}$$

Then as $t \to -\infty$ we have

$$\widetilde{\alpha}(t) = \widetilde{\alpha}_0 e^{\alpha t} + O(e^{2\alpha t}), \tag{3.526}$$

$$\widetilde{\alpha}_0 > 0 \textit{ is given explicitly by (3.569) and is of the form } \vec{Const} \cdot W. \tag{3.527}$$

Moreover,

$$\widetilde{U}(t) = U(t) - \mathcal{U}(\infty) = \widetilde{U}_0 e^{\alpha t} + O(e^{2\alpha t}), \tag{3.528}$$

where $\widetilde{U}_0$ is given explicitly in (3.575) and is of the form $Const \cdot W$. Then $\widetilde{\gamma}_0 = \widetilde{U}_0(1)$.

(b) Define u^* as the solution to the nonlinear equation (3.414) with initial condition at $-\infty$

$$\lim_{t \to -\infty} e^{-\alpha t} u^*(t) = 1. \tag{3.529}$$

Then as $t \to -\infty$ we have the second order expansion:

$$u^*(t) = e^{\alpha t} - (b - \frac{\widetilde{\alpha}_0^*}{\alpha}) e^{2\alpha t} + O(e^{3\alpha t}), \tag{3.530}$$

where $\widetilde{\alpha}_0^*$ is obtained from $\widetilde{\alpha}_0$ by setting $W = 1$ in the formula (3.569). □

With these results we obtain immediately (3.386). This would complete the Proof of Proposition 3.10.

It remains to prove Lemma 3.15. To prove this we need some preparation we do in the next part.

Part 5 - Statement of proof of Lemma 3.16

We and formulate a statement that gives an explicit representation of u^* (and hence of the limit of u^N suitably shifted in terms of $\alpha(\cdot)$ which is exact up to the third order error terms:

Lemma 3.16 *(Identification of u in terms of $\widehat{u}$)*

Consider the solution to the ODE

$$\frac{du^*(t)}{dt} = \alpha(t)[u^*(t) - b(t)(u^*(t))^2], \quad -\infty < t < \infty, \tag{3.531}$$

with boundary condition at $-\infty$ given by $\lim_{t \to -\infty} e^{-\alpha t} u^*(t) = 1$ and with the abbreviations

$$b(t) = 1 + \frac{\gamma(t)}{\alpha(t)}. \tag{3.532}$$

(Recall that the general case is obtained by a time shift by $\frac{\log W}{\alpha}$ if 1 is replaced by W and $\alpha(\cdot), b(\cdot)$ are shifted accordingly.)

Define with $b = 1 + (\gamma/\alpha)$ the function $\widehat{u}$ on $\mathbb{R}$:

$$\widehat{u}(t) = \frac{e^{\alpha t} e^{\int_{-\infty}^t (\alpha(s) - \alpha) ds}}{1 + b e^{\alpha t} e^{\int_{-\infty}^t (\alpha(s) - \alpha) ds}}, \quad -\infty < t < \infty. \tag{3.533}$$

(a) Then

$$|u^*(t) - \widehat{u}(t)| = O(e^{3\alpha t}). \tag{3.534}$$

The function $\widehat{u}$ satisfies (here again $b = 1 + \frac{\gamma}{\alpha}$)

$$\widehat{u}(t) \sim \frac{e^{\alpha t} e^{\int_{-\infty}^t (\alpha(s) - \alpha) ds}}{1 + b e^{\alpha t}} \text{ as } t \to -\infty. \tag{3.535}$$

(b) We have for $T_N = \alpha^{-1} \log N$ that in distribution:

$$\frac{1}{N} u^N(T_N + t) \underset{N \to \infty}{\Longrightarrow} u^*(t + \frac{\log W}{\alpha}) = \widehat{u}(t + \frac{\log W}{\alpha}) + O(e^{3\alpha t}). \quad \square \tag{3.536}$$

Part 6 - Proof of Lemma 3.16

(a) We start with the following observation. Since by the coloured particle calculation we obtained earlier in this proof, see (3.516), (3.517),

$$(3.537)\qquad |\gamma(t)-\gamma|+|\alpha(t)-\alpha|\le \text{const}\cdot e^{\alpha t}$$

and $\alpha>0$, then

$$(3.538)\qquad |b(t)-b|=|b(t)-(1+\frac{\gamma}{\alpha})|\le \text{const}\cdot e^{\alpha t}.$$

When $b(\cdot)$ is not constant we cannot obtain a closed form solution of (3.531). However we can obtain an approximation that describes the asymptotics as $t\to-\infty$ accurate up to terms of order $O(e^{2\alpha t})$ as follows.

Let

$$(3.539)\qquad (\widehat{u}_{t_0}(t))_{t\in\mathbb{R}}$$

denote the solution of the modified equation (3.531) where we replace our *function* $b(t)$ *by the constant* b and put the initial condition u_{t_0} at time t_0.

The solution of the modified equation is given by the formula

$$(3.540)\qquad \widehat{u}_{t_0}(t)=\frac{\widehat{u}_{t_0}(t_0)e^{\int_{t_0}^t\alpha(s)ds}}{1+b\widehat{u}_{t_0}(t_0)(e^{\int_{t_0}^t\alpha(s)ds}-1)},\quad t\ge t_0.$$

Now let

$$(3.541)\qquad u_{t_0}=e^{\alpha t_0},$$

so that

$$(3.542)\qquad \widehat{u}_{t_0}(t)=\frac{e^{\alpha t}e^{\int_{t_0}^t(\alpha(s)-\alpha)ds}}{1+be^{\alpha t}e^{\int_{t_0}^t(\alpha(s)-\alpha)ds}-be^{\alpha t_0}},\quad t\ge t_0.$$

We then get with (3.535) that

$$(3.543)\qquad \widehat{u}(t)=\lim_{t_0\to-\infty}\widehat{u}_{t_0}(t)=\frac{e^{\alpha t}e^{\int_{-\infty}^t(\alpha(s)-\alpha)ds}}{1+be^{\alpha t}e^{\int_{-\infty}^t(\alpha(s)-\alpha)ds}},\quad -\infty<t<\infty,$$

noting that the integrals are well defined by (3.537). Hence $\widehat{u}$ satisfies a differential equation ((3.531) with $b(t)\equiv b$) which we now use to estimate $\widehat{u}-u^*$.

Let $v(t):=(\widehat{u}(t)-u^*(t))$. Then $v(t)$ satisfies

$$(3.544)\qquad \frac{dv(t)}{dt}=\widetilde{\alpha}(t)v(t)-b(\widehat{u}(t)+u^*(t))v(t)+(b(t)-b)(u^*(t))^2.$$

Using (3.537), (3.538) and the fact that $u^*(t)\le\text{const}\cdot e^{\alpha t}$ we then obtain that

$$(3.545)\qquad \frac{dv(t)}{dt}\le\text{const}\cdot e^{\alpha t}v(t)+O(e^{3\alpha t}).$$

Therefore $|v(t)|\le\text{const}\cdot e^{3\alpha t}$ and hence

$$(3.546)\qquad |u^*(t)-\widehat{u}(t)|\le\text{const}\cdot e^{3\alpha t}\text{ as }t\to-\infty.$$

We conclude with an estimate for the integral term in $\widehat{u}$

$$(3.547)\qquad \begin{aligned}&(\alpha(t)+\gamma(t))\widehat{u}(t)\\ &= \frac{(\alpha(t)+\gamma(t))e^{\alpha t}e^{\int_{-\infty}^{t}(\alpha(s)-\alpha)ds}}{1+b(e^{\alpha t}e^{\int_{-\infty}^{t}(\alpha(s)-\alpha)ds})}.\end{aligned}$$

Furthermore by (3.524),

$$(3.548)\qquad (\alpha(t)+\gamma(t))u^*(t) \le (\alpha+\gamma)e^{\alpha t}.$$

Then solving in (3.547) for the $e^{\int}$-term we obtain with (3.534) for $t \to -\infty$ the relation:

$$(3.549)\qquad I(t) = e^{\int_{-\infty}^{t}(\alpha(s)-\alpha)ds} \le \frac{1}{1-be^{\alpha t}} = 1 + be^{\alpha t} + o(e^{\alpha t}).$$

(b) The second assertion follows from part (a). The first assertion follows from (3.312). This completes the proof of Lemma 3.16.

Part 6 - Proof of Lemma 3.15

We separately show the different claimed relations in points (1)-(3) for $\widetilde{\alpha}, \widetilde{\gamma}$ and then u^*.

(1) Relation (3.526) and (3.527)

Here our strategy is to express $\widetilde{\alpha}(\cdot)$ in terms of the rewritten (u, U) using the Markov process generated by Q_0^α and the operators Q_1 and L. Using (3.422), and (3.572) we can rewrite the nonlinear system defining (u, U) in the following form. Let $U(t), Q_0^a, Q_1, L$ be defined as in (3.415), (3.417), (3.418) and define ν by

$$(3.550)\qquad \nu(j) = j1_{j>1}.$$

Furthermore recall that by setting $a = \alpha$ we get the following:

$$(3.551)\qquad (S_t^\alpha)_{t\ge 0} = \text{semigroup with generator } Q^\alpha = Q_0^\alpha + \alpha 1_{j=1}.$$

This semigroup has a unique entrance law from $t \to -\infty$, since it is standard to verify that S^α defines an ergodic Markov process.

Next let $(U^\alpha(t))_{t\in\mathbb{R}}$ solve the (forward) equation for generator Q^α, i.e.

$$(3.552)\qquad \frac{\partial}{\partial t}U^\alpha(t) = U^\alpha(t)Q^\alpha, \quad t \in \mathbb{R},$$

and define the difference process $(\widetilde{U}(t))_{t\in\mathbb{R}}$ (we now suppress the superscript α in $\widetilde{U}$):

$$(3.553)\qquad \widetilde{U}(t) = U(t) - U^\alpha(t).$$

Then

$$(3.554)\qquad \frac{\partial \widetilde{U}(t)}{\partial t} = \widetilde{U}(t)Q_0^\alpha + \widetilde{\alpha}(t)[U(t)Q_1 + 1_{j=1}] + u(t)(\alpha(t)+\gamma(t))[U(t)L - 1_{j=1}].$$

We can now represent due to the definition of $\alpha(\cdot), U, U^\alpha, \widetilde{\alpha}(\cdot)$ the function $\alpha(t)$ as

$$(3.555)\qquad \alpha(t) - \alpha =< c\widetilde{U}(t), \nu > .$$

Therefore (recall (3.550)) by the formula of partial integration for semigroups (for ν see (3.550) with the semigroup S^α of U^α as the reference semigroup and then U as the wanted object:

$$(3.556)\quad \begin{aligned}\widetilde{\alpha}(t) &= \alpha(t)-\alpha = \langle c\widetilde{U}(t),\nu\rangle\\ &= \int_{-\infty}^{t} c[\widetilde{\alpha}(s)\langle (U(s)Q_1+1_{j=1})S^{\alpha}_{t-s},\nu\rangle + u(s)(\alpha+\gamma)\langle (U(s)L-1_{j=1})S^{\alpha}_{t-s},\nu\rangle]ds.\end{aligned}$$

Next set

$$(3.557)\quad \mathcal{U}(\infty)=(p_1,p_2,\dots)$$

and note that by explicit calculation (see (3.418), (3.419)):

$$(3.558)\quad \mathcal{U}(\infty)Q_1+1_{j=1}=(1-p_1,-p_2,-p_3,\dots),\quad \mathcal{U}(\infty)L-1_{j=1}=(-1,p_1,p_2,\dots).$$

Note that for $t\to-\infty$:

$$(3.559)\quad \lim_{t\to-\infty} U(t)=\mathcal{U}(\infty),\qquad \|\mathcal{U}(\infty)-U(t)\|_1=O(e^{\alpha t}),$$

where $\mathcal{U}(\infty)$ is the stable size distribution of the McKean-Vlasow dual process where the norm $\|\cdot\|_1$ is as in (3.263). Hence we get from (3.556) that

$$\begin{aligned}\widetilde{\alpha}(t) \quad &= \int_{-\infty}^{t} c[\widetilde{\alpha}(s)\langle (\mathcal{U}(\infty)Q_1+1_{j=1})S^{\alpha}_{t-s},\nu\rangle\\ &+u(s)(\alpha+\gamma)\langle(\mathcal{U}(\infty)L-1_{j=1})S^{\alpha}_{t-s},\nu\rangle]ds\\ &+O(e^{2\alpha t}).\end{aligned}$$

Next note that (recall (3.550) and (3.558), (3.557) and the fact that the Markov process for S^{α} increases from the initial value 1 stochastically to its equilibrium) we have the relations:

$$(3.560)\quad -\alpha<\langle(\mathcal{U}(\infty)Q_1+1_{j=1})S^{\alpha}_t,\nu\rangle<0 \text{ and } \langle(\mathcal{U}(\infty)L-1_{j=1})S^{\alpha}_t,\nu\rangle>0.$$

Recall furthermore that

$$(3.561)\quad \langle\mathcal{U}(\infty)S^{\alpha}_t,\nu\rangle=\frac{\alpha}{c} \text{ for all } t\geq 0.$$

Let now (use (3.560) for possitivity):

$$(3.562)\quad A_1=c\int_0^{\infty}\langle(\mathcal{U}(\infty)L-1_{j=1})e^{-\alpha s}S^{\alpha}_s,\nu\rangle ds>0,$$

$$(3.563)\quad A_2=-c\int_0^{\infty} e^{-\alpha s}\langle(\mathcal{U}(\infty)Q_1+1_{j=1})S^{\alpha}_s,\nu\rangle ds>0.$$

Now multiply through both sides of (3.556) by $\alpha e^{-\alpha t}$ and set

$$(3.564)\quad \widehat{\alpha}(t)=e^{-\alpha t}\widetilde{\alpha}(t),\quad \widehat{S}^{\alpha}_t=e^{-\alpha t}S^{\alpha}_t.$$

We obtain the equation:

$$(3.565)\quad \begin{aligned}\widehat{\alpha}(t) \quad = \quad c\int_{-\infty}^{t} &[\widehat{\alpha}(s)\langle(\mathcal{U}(\infty)Q_1+1_{j=1})\widehat{S}^{\alpha}_{t-s},\nu\rangle\\ &+\widehat{u}(s)(\alpha+\gamma)\langle(\mathcal{U}(\infty)L-1_{j=1})\widehat{S}^{\alpha}_{t-s},\nu\rangle]ds.\end{aligned}$$

This is a *renewal-type equation* of the form (choosing suitable positive functions f,g, recall (3.560)):

$$(3.566)\quad \widehat{\alpha}=\widehat{\alpha}*(-f)+\widehat{u}*g,$$

where the integral over f over $\mathbb{R}$ is equal to A_2, which is less than 1 due to relation (3.561) and (3.564), furthermore $\widehat{u}(t) \to W$ as $t \to -\infty$ and g is an integrable positive function with integral A_1.

We now claim that as $t \to -\infty$

$$\widehat{\alpha}(t) \longrightarrow C \text{ and } |\widehat{\alpha}(t) - C| = O(e^{\alpha t}), \tag{3.567}$$

where C is calculated as usual in renewal theory.

Then the solution (uniqueness is verified below) to (3.566) is given by

$$\widetilde{\alpha}(t) = \widetilde{\alpha}_0 e^{\alpha t} + O(e^{2\alpha t}), \tag{3.568}$$

where $\widetilde{\alpha}_0 > 0$ is given by

$$\widetilde{\alpha}_0 = \frac{A_1}{1 + A_2} W. \tag{3.569}$$

The fact that the errorterm in (3.568) is of the form $O(e^{2\alpha t})$ follows from (3.560). Moreover (3.568) is the unique solution of order $O(e^{\alpha t})$ to (3.556). To verify the uniqueness we argue as follows.

The difference of two solutions h_1, h_2 must solve according to (3.566) that

$$(h_1 - h_2) = (h_1 - h_2) * (-f). \tag{3.570}$$

We can then verify that either $h_1 \equiv h_2$ or

$$|(h_1 - h_2)(t)| < \sup_{(-\infty,0)} |h_1(t) - h_2(t)| \tag{3.571}$$

and therefore indeed $h_1 - h_2 \equiv 0$.

This completes the proof of (3.526) and (3.527).

Remark 38 *We have with (3.246) and (3.288) the following ODE for $\alpha(t)$ as function of (u, U):*

$$\begin{aligned} \frac{\partial \alpha(t)}{\partial t} = {} & \frac{\partial}{\partial t}[c \sum_{j=2}^{\infty} jU(t,j)] = c \sum_{j=2}^{\infty} j(\frac{\partial}{\partial t} U(t,j)) \\ = {} & c \sum_{j=2}^{\infty} j \Big\{ [\alpha(t)(1-u(t)) - \gamma(t)u(t)] \, 1_{j=1} \\ & + u(t)(\alpha(t)+\gamma(t))[1_{j\neq 1} U(t,j-1) - U(t,j)] \\ & + c(j+1)U(t,j+1) - cjU(t,j) 1_{j \neq 1} \\ & + s(j-1) 1_{j\neq 1} U(t,j-1) - sjU(t,j) \\ & + \frac{d}{2}(j+1)jU(t,j+1) - \frac{d}{2} j(j-1)) 1_{j\neq 1} U(t,j) \\ & - \Big(\alpha(t)(1-b(t)u(t))\Big) \cdot U(t,j) \Big\}. \end{aligned} \tag{3.572}$$

Collecting terms we obtain:

$$\begin{aligned} \frac{\partial \alpha(t)}{\partial t} & = (s - \alpha(t) + \frac{d}{2})(\alpha(t)+\gamma(t)) - \frac{cd}{2} \sum_{j=1}^{\infty} j^2 U(t,j) \\ & + u(t)(\alpha(t)+\gamma(t))^2 + cu(t)(\alpha(t)+\gamma(t)). \end{aligned} \tag{3.573}$$

Hence we can determine α as solution of an ODE once we are given (u, U).

(2) Relations (3.528) and (3.385).

Recall (3.554) which yields

$$\frac{\partial \widetilde{U}(t)}{\partial t} = \widetilde{U}(t)Q_0^\alpha + \widetilde{\alpha}(t)[\mathcal{U}(\infty)Q_1 + 1_{j=1}] + u(t)(\alpha+\gamma)[\mathcal{U}(\infty)L - 1_{j=1}] + O(e^{2\alpha t}). \tag{3.574}$$

Then using the expression (3.568) for $\widetilde{\alpha}(t)$ and $u(t) = We^{\alpha t} + O(e^{2\alpha t})$ we obtain

$$\widetilde{U}_0 = \int_{-\infty}^0 e^{\alpha s} S_{-s}^\alpha V ds + O(e^{2\alpha t}) \tag{3.575}$$

where $V = W(\widetilde{\alpha}_0^*[\mathcal{U}(\infty)Q_1 + 1_{j=1}] + (\alpha+\gamma)[\mathcal{U}(\infty)L - 1_{j=1}])$. Recalling that S_t^α is a semigroup on $\mathcal{P}(\mathbb{N})$, it follows that the integral is well-defined.

(3) Relation (3.530).

With the knowledge we have now the expression (3.530) then follows by (3.534) by substitution with (3.568) in (3.535).

3.2.10 Weighted occupation time for the dual process

In this subsubsection we focus on the behaviour of the quantity $\Xi_N(\alpha^{-1}\log N + t, 2)$ in the limit $N \to \infty$ as a function of t. We know that this limit is $\mathcal{L}_t(2)$ which is a random probability measure on $[0,1]$, in order to identify its distribution it suffices to compute the moments via duality and (3.320). Recall also that the weighted occupation time of the dual determines the probability that no mutation occurred changing the value at the tagged site from type 1 to 2. Why, what do we have to prove about the dual process? This we now explain first.

Consider $T_N = \alpha^{-1}\log N$ and $t_0(N) \uparrow \infty, \quad t_0(N) = o(\log N)$. Recall that

$$\Pi_s^{N,k,l} = \sum_{i_w, i_R, i_P} (i_W + i_R + i_P)\Psi_s^N(i_W, i_R, i_P), \tag{3.576}$$

where $\Pi_s^{N,k,l}$ is given by the dual particle system started with k particles at each of ℓ sites.

Then we know that we only have to replace W by $W^{(k,\ell)}$ and otherwise we get the same equation as we had for $k=\ell=1$. Hence we have

$$\int_{t_0(N)}^{(t+T_N)} [N^{-1}\Pi_s^{N,k,\ell}]ds \sim \frac{1}{c}\int_{-\infty}^t (\alpha^{(k,\ell)}(s) + \gamma^{(k,\ell)}(s))u^{(k,\ell)}(s)ds, \text{ as } N \to \infty \tag{3.577}$$

and we need the r.h.s. in the limit as $t \to -\infty$. Integrating the expression given by (3.386) we get the following expansion:

$$\int_{-\infty}^t (\alpha(k,\ell)(s) + \gamma(k,\ell)(s))u(k,\ell)(s)ds = W^{(k,\ell)}\frac{(\alpha+\gamma)}{\alpha}e^{\alpha t} - \frac{\kappa^*}{2\alpha}(W^{(k,\ell)})^2 e^{2\alpha t} + O(e^{3\alpha t}). \tag{3.578}$$

Recall that in terms of the *normalized* solution of the equation for the dual (u^*, U^*), (i.e. $e^{\alpha t}u^*(t) \to 1$ as $t \to -\infty$) we have by the above on the one hand

$$\begin{aligned}\mathcal{L}\left\{[\tfrac{1}{N}\int_{t_0(N)}^{T_N+t} \Pi_u^{N,k,\ell}du]\right\} &\underset{N\to\infty}{\Longrightarrow} \mathcal{L}\left\{\int_{-\infty}^{t+\frac{\log W^{k,\ell}}{\alpha}} \tfrac{1}{c}(\alpha(s)+\gamma(s))u^*(s)ds\right\} \\ &=: \nu_{k,\ell}(t) \in \mathcal{P}([0,\infty))\end{aligned} \tag{3.579}$$

and on the other hand in terms of the original process we have (cf. (3.118), (3.597)),

$$E\left(\left[\int_0^1 x^k \mathcal{L}_t(2,dx)\right]^\ell\right) = \int_0^\infty e^{-my}\nu_{k,\ell}(t,dy) \text{ for all } k,\ell \in \mathbb{N}. \tag{3.580}$$

Note that the random time shift by $\frac{\log W^{k,\ell}}{\alpha}$ plays an essential role. The interplay between this shift and the nonlinearity of $u(t)$ determines the distribution of $\mathcal{L}_t(2)$. To determine this effect we study the properties of the r.h.s. in the two equations above. In particular we want to show that $\mathcal{L}_t(2)$ is neither δ_0 nor δ_1 and we want to show in fact that it is truly random and has its mass on $(0,1)$. This can be translated into properties of the r.h.s. of (3.580) which we will study using the dual.

We next collect some key facts needed to calculate the probability of mutation jumps as $N \to \infty$ which appear on the r.h.s. of (3.580).

Lemma 3.17 *(Properties limiting dual occupation density)*
(a) If $c > 0$ *the limit object* $\nu_{k,\ell}$ *satisfies:*

$$\nu_{k,\ell}(t,(0,\infty)) = 1, \tag{3.581}$$

and

$$\lim_{n\to\infty} \nu_{n,1}(t,[K,\infty)) = 1, \text{ for all } K. \tag{3.582}$$

(b) Denote by $<$ *strict stochastic order of probability measures. Then for* $c>0, d>0$*:*

$$\nu_{k+1,1} > \nu_{k,1} \tag{3.583}$$

$$\nu_{1,2}(t) < \nu_{1,1}(t) \star \nu_{1,1}(t), \tag{3.584}$$

where $\star$ *denotes convolution.*
(c) Let $n_0 \in \mathbb{N}$ *and set*

$$t_{n_0}(\varepsilon, N) := \inf\{t : \Pi_t^{N,n_0,1} = \lfloor \varepsilon N \rfloor\} - \frac{1}{\alpha}\log N. \tag{3.585}$$

Then every weak limit point (in law) as $N \to \infty$*, denoted* $t_{n_0}(\varepsilon)$ *satisfies:*

$$t_{n_0}(\varepsilon) \to -\infty \text{ in probability as } n_0 \to \infty. \tag{3.586}$$

Furthermore

$$t_{n_0}(\varepsilon) \longrightarrow -\infty \text{ in probability as } \varepsilon \to 0. \qquad \square \tag{3.587}$$

Proof of Lemma 3.17

Proof of (a). The claim $\nu_{k,\ell}(t,\{0\}) = 0$ follows from (3.381) since $W^{k,\ell} > 0$ a.s. and $\nu_{k,\ell}(t,(\{\infty\})) = 0$ since for $T_N = \alpha^{-1}\log N$ and every $t \in \mathbb{R}$ we have:

$$\limsup_{N\to\infty} E\left[\frac{1}{N}\int_0^{T_N+t} \Pi_u^{N,k,\ell} du\right] < \infty. \tag{3.588}$$

Therefore $\nu_{k,\ell}$ is a probability measure concentrated on $(0,\infty)$.
The relation (3.582) will follow from the argument given for the proof of (3.586) in (c).

Proof of (b). The first part of (b) is immediate by a coupling argument realising the $(k+1)$ and k particle system on one probability space. The second part follows from the fact two typical individuals picked among the descendants of the two populations interact through coalescence with

positive probability before they jump since once they both occupy $O(N)$ sites, so that we have a fraction of both populations overlapping. This proves (b).

Proof of (c). The main idea in the proof of (c) is that, asymptotically as $n_0 \to \infty$, starting n_0 particles at a tagged site the number of particles at this site decreases to $O(1)$ in time 1 but during this time still produces $O(\log n_0)$ migrants as we shall see below. Therefore if the number of initial particles, n_0 increases to infinity, then an increasing number of populations start growing like $W_i e^{\alpha t}$ and they do so independently. Hence writing the total population as

$$(3.589) \quad \sum_{i=1}^{m(n_0)} e^{\alpha(t+\alpha^{-1}\log W_i)},$$

we see that if $m(n_0)$ diverges as $n_0 \to \infty$, then the total number of particles reaches $O(N)$ at a time $(\frac{1}{\alpha}\log N + t_{n_0})$ where t_{n_0} decreases as n_0 increases to $+\infty$. Hence it remains to give the formal argument for the fact that $m(n_0)$ is of order $\log n_0$ as $n_0 \to \infty$, which runs as follows.

Consider Kingman's coalescent $(C_t)_{t\geq 0}$ starting with countably many particles . Consider the number of particles in a spatial Kingman coalescent which jump before coalescing at the starting site. This is the given via the rate of divergence of the entrance law of Kingman's coalescence from 0. Hence we need the rate of divergence

$$(3.590) \quad \int_\delta^1 |C_t| dt \sim |\log \delta| \text{ as } \delta \to 0,$$

which follows since the rate of divergence is wellknown to be δ^{-1}. We connect this to an initial state with n_0 particles which is obtained by restricting the entrance law of Kingman's coalescent. We then see that the number of migration jumps diverges as n_0 tends to infinity.

To get (3.587) we use that $t_{n_0}(\varepsilon, N)$ can be bounded from above by the time where $\Pi^{\infty,n_0,1}$ reached first $\lfloor \varepsilon N \rfloor$. By the analysis of (K_t, ζ_t) we know that for this it suffices to show that $\widetilde{\tau}^{N,n_0}(\varepsilon) \longrightarrow -\infty$ as $\varepsilon \to 0$, which follows from the exponential growth of K_t immediately.

This completes the proof of the lemma.

3.2.11 Proof of Proposition 1.12, Part 1: Convergence to limiting dynamics

In the proof of Proposition 1.7 it sufficed to work with the expected value $E[x_1^N(\frac{1}{\alpha}\log N + t)]$ (since $0 \leq x_1 \leq 1$). However in order to prove Proposition 1.12 it is necessary to work as well with higher moments in order to identify not only the time of emergence but to determine the dynamic of fixation. The strategy is to carry out the following two steps next:

1. show that weak convergence of $(\Xi_N^{\log,\alpha}(t))_{t\geq t_0}$ to the limiting deterministic McKean-Vlasov dynamics for $t \geq t_0$ follows from the weak convergence of the marginal distributions at t_0,

2. prove that the one dimensional marginals $(\Xi_N^{\log,\alpha}(t_0))$ converge weakly to a random (not deterministic) probability measure on $[0,1]$.

Step 1 Reduction to convergence of one-dimensional marginals.
In order to prove that $(\Xi_N^{\log,\alpha}(t))_{t\in\mathbb{R}}$ converges weakly to a random solution of the McKean-Vlasov equation we argue first that it suffices to prove that the one-dimensional marginals converge. Namely if one has that, then one can use the Skorohod representation to get a.s. convergence on some joint probability space. Then however we can use that (using duality to obtain a Feller property)

$$(3.591) \quad \Xi_N^{\log,\alpha}(t_0) \to \mu_0,$$

implies $\{\Xi_N^{\log,\alpha}(t)\}_{t_0\le t\le t_0+T}$ converges to a solution of the McKean Vlasov equation (1.18) with initial value μ_0.

This follows by noting that we claim here a standard McKean-Vlasov limit of an exchangeable system satisfying the martingale problem (1.3), (1.4) and with initial empirical measures converging. The proof is a modification of the standard proof of the McKean-Vlasov limit (for details see proof of Theorem 9, [DG99]).

Step 2 Convergence of marginals: reformulation in terms of the dual.
We now prove the required convergence of the one-dimensional marginal distributions. To do this we recall first that moments determine probability measures on $[0,1]$ and that therefore the collection of "moments of moments" of a random measure denoted X on $[0,1]$ given by

$$E[\prod_{i=1}^m \int x^{k_i} X(dx)],\ k_i, m\in\mathbb{N}, \tag{3.592}$$

determine the law. Compactness is automatic so therefore it suffices to verify that for all $k,\ell\in\mathbb{N}$ the (k,ℓ) moments of the mass of type 1

$$m^N_{k,\ell}(t) = E([\int_{[0,1]} x^k \Xi_N^{\log,\alpha}(t,1)(dx)]^\ell), \quad \forall k,\ell\in\mathbb{N}, \tag{3.593}$$

converge as $N\to\infty$ to conclude weak convergence of the one-dimensional marginal distributions.

In order to analyse the empirical measure and its functionals we return to the original system on the site space $\{1,2,\cdots,N\}$ and express the quantity through moments of observables of the system. These moments are obtained in terms of the dual process by considering the dual with initial function generated by taking the k-product given by $1_{\{1\}}\otimes\ldots\otimes 1_{\{1\}}$ at ℓ not necessarily distinct sites.

This representation can be simplified a bit. Namely note that for bounded exchangeable random variables $(X_1,\cdots,X_N)$ one has:

$$\lim_{N\to\infty} E[(\frac{1}{N}\sum_{i=1}^N X_i^k)^\ell] = E[\prod_{i_1\ne i_2\ne\ldots\ne i_\ell} X_{i_j}^k]. \tag{3.594}$$

Therefore to compute $\lim_{N\to\infty} E[\int_{[0,1]} x^k \Xi_N^{\log,\alpha}(t,1)(dx)]^\ell$ we use the dual process $(\eta_t,\mathcal{F}_t^+)_{t\ge0}$ where the particle system η starts in configuration η_0 with *k-particles* at *each of ℓ distinct sites* $i=1,\cdots,\ell$ and the function-valued part starts with

$$\mathcal{F}_0^+ = \bigotimes_{j=1}^{\ell}\bigotimes_{i=1}^{k}(1_{\{1\}}). \tag{3.595}$$

Let

$$\Pi_u^{N,k,\ell} \tag{3.596}$$

denote the resulting number of dual particles at time u.

Recall that the mutation jump to type 2 (from type 1) is given by $1_{\{1\}}\to 0$. Recall furthermore that the selection operator $1_{A_2}=1_{\{2\}}$ preserves the product form (note for every state f before the first mutation $1\to 2, f1_{(2)}\equiv 0$) and just generates another factor $1_{\{1\}}$ with a new variable. Therefore the dual $\mathcal{F}_t^+$ (before the first rare mutation $1\to 2$ occurs) is a product of factors $1_{\{1\}}$

and this product integrated with respect to $\mu_0^{\otimes}$ with $\mu_0(\{1\}) = 1$ is 0 or 1 depending on whether or not a mutation jump has occurred. Therefore

$$
\begin{aligned}
(3.597)\qquad m^N_{k,\ell}(t) &:= P[\text{no mutation jump } 1 \to 2 \text{ occurred by time } T_N + t] \\
&= E[\exp(-\frac{m}{N} \int_0^{T_N+t} \Pi_u^{N,k,\ell} du)].
\end{aligned}
$$

Hence in order to prove that the following limit exists:

$$(3.598)\qquad m_{k,\ell}(t) = \lim_{N\to\infty} m^N_{k,\ell}(t),$$

it suffices to prove that the following holds:

$$(3.599)\quad \mathcal{L}\left[\frac{1}{N}\int_0^{T_N+t} \Pi_u^{N,k,\ell} du\right] \underset{N\to\infty}{\Longrightarrow} \nu_{k,\ell}(t) \in \mathcal{P}([0,\infty]) \quad , \quad \nu_{k,\ell}(t)((0,\infty)) = 1, \text{ for every } k,\ell \in \mathbb{N}$$

and then we automatically have as well the formula

$$(3.600)\qquad m_{k,\ell}(t) = \int_0^\infty e^{-my} \nu_{k,\ell}(t)(dy).$$

The existence of the limit (3.599) is a result on the dual process which we proved in Proposition 3.8, part (3.320). This will be used in the next section to prove that we have convergence to a random McKean-Vlasov limiting dynamic which is truly random by showing that the limiting variance of the empirical mean is strictly positive.

3.2.12 Proof of Proposition 1.12, Part 2: The random initial growth constant

In the previous sections we have studied the asymptotics of the dual process in terms of the Crump-Mode-Jagers branching process and the pair (u, U) following a certain nonlinear equation. In this section we use these results to establish that the limiting empirical distribution of types is given by the McKean-Vlasov dynamics with random initial condition at time $-\infty$. Recall that in order to describe emergence we assume that initially only type 1 is present and we wish to determine the distribution of $\bar{x}_2(t) := \lim_{N\to\infty} \bar{x}_2^N(T_N + t)$ and its behaviour as $t \to -\infty$ subsequently.

We have so far established that $\Xi_N^{\log,\alpha}(t, 2)$ converges to $\mathcal{L}_t(2)$ which is a solution to the McKean-Vlasov dynamics and that

$$(3.601)\qquad \int_0^1 x\mathcal{L}_t(2)(dx) \to 0 \text{ as } t \to -\infty.$$

Therefore by Proposition 1.3 (see proof in Section 3.2.2), Proposition 1.7 and (1.85) the following limits exist in distribution and satisfy

$$(3.602)\qquad \lim_{t\to-\infty} e^{\alpha|t|} \int_0^1 x\mathcal{L}_t(dx) = \lim_{t\to-\infty}\lim_{N\to\infty} e^{\alpha|t|}\bar{x}_2^N(\frac{\log N}{\alpha} + t) =^* \mathcal{W},$$

so that $\mathcal{L}_t$ is a random shift of $\mathcal{L}^*$ (cf. (1.31)), namely, $\mathcal{L}^*_{t+\frac{\log {}^*\mathcal{W}}{\alpha}}$ and

$$(3.603)\qquad \mathcal{L}[{}^*\mathcal{W}] = \lim_{t\to-\infty} \mathcal{L}[e^{\alpha|t|}\int_0^1 x\mathcal{L}_t(dx)].$$

We next consider the first and second moments of ${}^*\mathcal{W}$.

Proposition 3.18 *(Moments of ${}^*\mathcal{W}$)*
The first and second moments of ${}^*\mathcal{W}$ satisfy:

$$E[{}^*\mathcal{W}] = m^* E[W], \text{ with } \quad m^* = \frac{mb}{c}, \quad b = (1 + \frac{\gamma}{\alpha}), \tag{3.604}$$

$$Var\Big[\int_0^1 x\mathcal{L}_t(2)(dx)\Big] = O(e^{2\alpha t}), \tag{3.605}$$

and

$$Var[{}^*\mathcal{W}] = m^* \frac{\kappa^*}{2\alpha}(E[W])^2. \qquad \square \tag{3.606}$$

Proof of Proposition 3.18 Recall the equation (3.578) from Subsubsection 3.2.10 and note that this approximates $(u(t))_{t\in\mathbb{R}}$ for $t \to -\infty$ up to terms $O(e^{3\alpha t})$:

$$\int_{-\infty}^t \frac{1}{c}(\alpha(s) + \gamma(s))u(s)ds = \frac{1}{c}\left(\frac{(\alpha+\gamma)}{\alpha} We^{\alpha t} - \frac{\kappa^*}{2\alpha} W^2 e^{2\alpha t}\right) + O(e^{3\alpha t}), \quad \kappa^* > 0. \tag{3.607}$$

This follows from Proposition 3.10 - see Lemma 3.15 for the proof that $\kappa^* > 0$.

Therefore

$$\begin{aligned} &\exp\left(-\frac{m}{c}\int_{-\infty}^t (\alpha(s) + \gamma(s))u(s)ds\right) \\ &= \exp\left(-m^*\left(We^{\alpha t} - \frac{\kappa^*}{2\alpha}W^2 e^{2\alpha t}\right) + O(e^{3\alpha t})\right), \end{aligned} \tag{3.608}$$

where $m^* = \frac{mb}{c}$.

An immediate consequence of this relation is that (recall (3.598) for a definition):

$$m_{1,1}(t) = 1 - m^* E[W] e^{\alpha t} + O(e^{2\alpha t}), \text{ for } t \to -\infty \tag{3.609}$$

and therefore

$$\lim_{t\to-\infty} e^{\alpha|t|} E[\int_0^1 x\mathcal{L}_t(2)(dx)] = m^* E[W]. \tag{3.610}$$

Now consider the second moment of the type two mass at time t which we denote $m^{(2)}(t)$. Here we consider the dual process with one particle at each of two distinct sites. The corresponding growing clouds have independent random growth constants W_1 and W_2. We use the formula

$$\begin{aligned} m^{(2)}(t) := E[\int x\mathcal{L}_t(2)(dx)]^2 \quad &= E[\int x\mathcal{L}_t(1)(dx)]^2 \\ &-2E[\int x\mathcal{L}_t(1)(dx)] + 1 = m_{1,2}(t) - 2m_{1,1}(t) + 1. \end{aligned} \tag{3.611}$$

In order to calculate $m_{1,2}(t), m_{1,1}(t)$ we expand (3.608) to terms of order $o(e^{2\alpha t})$ for $\widetilde{W} = W_1 + W_2$ respectively W_1 where W_1, W_2 are independent copies of W, to obtain

$$\begin{aligned} &\exp\left(-m^*\left(b\widetilde{W}e^{\alpha t} - \frac{\kappa^*}{2\alpha}\widetilde{W}^2 e^{2\alpha t}\right) + O(e^{3\alpha t})\right) \\ &= 1 - m^*\left(b\widetilde{W}e^{\alpha t} - \frac{\kappa^*}{2\alpha}\widetilde{W}^2 e^{2\alpha t}\right) + \frac{1}{2}\left(m^* bWe^{\alpha t}\right)^2 + O(e^{3\alpha t}). \end{aligned} \tag{3.612}$$

Therefore

$$(3.613)\quad m_{1,2}(t) = 1 - m^*bE[W_1+W_2]e^{\alpha t} + m^*\frac{\kappa^*}{2\alpha}E[W_1+W_2]^2e^{2\alpha t} + \frac{1}{2}(m^*b)^2E[W_1+W_2]^2e^{2\alpha t} + O(e^{3\alpha t}).$$

Combining this with (3.609) and (3.611), we see that:

$$(3.614)\quad m^{(2)}(t) = m^*\frac{\kappa^*}{2\alpha}E[W_1]E[W_2]e^{2\alpha t} + \frac{1}{2}(m^*b)^2E[W_1]E[W_2]e^{2\alpha t} + O(e^{3\alpha t}).$$

Note that the coefficients of the higher order terms involve higher moments of W and cancel provided that the latter are finite. Hence to justify the cancellation it is necessary to verify that the higher moments of W are finite. However this was established in Lemma 3.4 in Subsubsection 3.2.5.

Inserting (3.609) and (3.614) in $Var[\bar{x}_2(t)] = m^2(t) - (E[\bar{x}_2(t)]^2$ we get:

$$(3.615)\quad \mathrm{Var}[\bar{x}_2(t)] = m^*\frac{\kappa^*}{2\alpha}b(E[W])^2e^{2\alpha t} + O(e^{3\alpha t}).$$

If we can show that

$$(3.616)\quad E[(\bar{x}_2(t))^3] = O(e^{3\alpha t}),$$

we get via normalizing by $e^{\alpha t}$ and letting $t \to -\infty$ indeed:

$$(3.617)\quad \mathrm{Var}[^*\mathcal{W}] = m^*\frac{\kappa^*}{2\alpha}b(E[W])^2.$$

Therefore we get (3.606) once we have the bound on the supremum over time of the third moment of $\bar{x}_2(t)$, which follows from Lemma 3.32 in Subsection 3.5.

Here we can use the fact that the collision-free regime gives a stochastic upper bound of the finite N system and then that in fact according to (3.617) we can bound the third moment of the scaled variable in t.

Remark 39 *The randomness expressed in $^*\mathcal{W}$ arises as a result of the fact that the mutation jump occurs in this time scale at an exponential waiting time (in t) together with the nonlinearity in (3.248), (3.249). The random variable W governing the initial growth of the dual population (which describes the limiting behaviour of the CMJ process) arise from the initial birth events in the dual process and has its main influence due to the nonlinearity of the evolution in t. However W will enter only through its mean in the law of $^*\mathcal{W}$ due to a law of large number effects.*

*The moments of $^*\mathcal{W}$ are determined in terms of the random variable W arising from the growth of K_t in the dual process. However as we have seen in the second moment calculation above the determination of the coefficients of $e^{k\alpha t}$ depend on an analysis of an asymptotic expansion of the nonlinear system and in particular $u(t)$ in orders up to k. Although we do not attempt to carry this out here, we show that with respect to Laplace transform order the law of $^*\mathcal{W}$ lies between the case in which the collisions are suppressed (deterministic case) and a modified system in which the the correction involving $\int(\alpha(s)-\alpha)ds$ is suppressed, that is, replacing $\alpha(s)$ by α. This illustrates the role of the collisions (nonlinearity) in producing the randomness in $^*\mathcal{W}$ and allows us to obtain bounds for the expected time to reach a small level ε (cf. [**?**]). This we pursue further in Subsection 3.4.*

3.2.13 Completion of the Proof of Proposition 1.12

We now collect all the pieces needed to prove all the assertions of the proposition, we proceed stepwise.

Step 1 Completion of the Proof of Proposition 1.12(a)

This was proved in Subsubsection 3.2.11 which heavily used 3.2.10.

Step 2 Completion of the proof of the Proposition 1.12(b)

In order to verify that the limiting marginal random probability measure $\mathcal{L}_t(j)$ has for $j = 1, 2$ really a nontrivial distribution and is not just deterministic, it suffices to show that

$$E([\int x\mathcal{L}_t(1)(dx)]^2) > (E([\int x\mathcal{L}_t(1)(dx)]))^2. \tag{3.618}$$

The quantity $\int x\mathcal{L}_t(1)(dx)$ arises as the limit in distribution of $\bar{x}_1^N(t)$, the empirical mean mass of type 1. Note that the empirical mean mass process $(\bar{x}_\ell^N(t); \ell = 1, 2)_{t\geq 0}$ is a random process which is not Markov! To see this look at the second moment of mean mass, it involves the covariance between two sites! More generally the dual representation of the kth moment of the empirical mean is given by starting k particles at k distinct sites.

Note that no coalescence occurs until time $O(\log N)$ and then the two clouds descending from the two different initial dual particles *interact* nontrivially. Then it is easy to conclude that the limiting variance of $\bar{x}_2^N(T_N + t)$ as $N \to \infty$ is not zero using (3.584).

Step 3 Completion of the proof of Proposition 1.12 (c).

The relations (1.88), (1.89) follow from the dual representation (3.580) of $\int_0^1 x^k\mathcal{L}_t(2)dx$ from the fact that $\nu_{k,\ell}(t,\cdot) \Longrightarrow \delta_0$ as $t \to -\infty$ and $\nu_{k,\ell}(t,\cdot) \Longrightarrow \delta_\infty$ as $t \to \infty$. This follows from combining (3.318) and (3.320) and then using (3.316) to get the first claim and using $u^{k,\ell}(t) \to \infty$ as $t \to \infty$ to get the second claim. For the proof of (1.91) we will use the general fact that

$$P(\mathcal{L}_t(1)(\{1\}) = 1) = \lim_{k\to\infty} m_{k,1}(t). \tag{3.619}$$

The result then follows from Lemma 3.17 equation (3.582).

Step 4 Completion of the proof of Proposition 1.12 (e)

The convergence of the empirical measure processes to a random solution of the McKean-Vlasov equation follows from Proposition 3.8 together with Proposition 1.12 (b).

Step 5 Proof of Proposition 1.12(f)

The claims (1.92)- (1.94) was proved in Subsubsection 3.2.12 in (3.602) and (3.603). The assertion (1.95) follows from Proposition 3.18.

Step 6 Proof of Proposition 1.12 (g)

For the assertion (1.96) we refer to a calculation we do later, see (3.888).

All these steps complete the proof of Proposition 1.12.

Remark 40 *The same approach can be carried out on Ω_L for fixed L or $\mathbb{Z}^d$. Namely the dual can be viewed as follows. Start with one particle, then the particle system (factors) can be viewed as a stochastic Fisher-KPP equation (at least until the time of the first mutation). In particular in $\mathbb{Z}^1$ we expect linear growth according to a travelling wave solution (see [CD]) and not exponential growth. In this case emergence occurs in time $O(\sqrt{N})$.*

3.3 Droplet formation: Proofs of Proposition 1.9-1.11

In this section we assume that $c > 0,\ m > 0,\ s > 0$, and examine the process in which rare mutants first appear in a finite time horizon in the population at the microscopic level, that is only at some rare sites they appear at a substantial level, then later on after large times these rare sites develop into growing *mutant droplets*, that is, a growing collection of sites occupied by the mutant type but still being of a total size $o(N)$. Then finally in even much larger times this leads to emergence at the macroscopic level once we have $O(N)$ sites in the droplet.

We show that in the microscopic growth regime of the droplet the type-2 mass is described by a population growth process with Malthusian parameter α and random factor $\mathcal{W}^*$. In the next Section 3.4 we use this structure to investigate the relation between ${}^*\mathcal{W}$ and $\mathcal{W}^*$. Recall that ${}^*\mathcal{W}$ arose in the context of emergence and fixation by considering time $\alpha^{-1}\log N + t$ and letting first $N \to \infty$ and then $t \to -\infty$ whereas $\mathcal{W}^*$ arises from the droplet growth at times t and by letting first $N \to \infty$ and then $t \to \infty$.

To carry out the analysis for the droplet growth we consider the following four time regimes for our population model:

$$[0, T_0),\ \text{where } 0 < T_0 < \infty, \tag{3.620}$$

$$[T_0, T_N),\ \text{where } T_N \to \infty,\ \text{as } N \to \infty \text{ and } T_N = o(\log N), \tag{3.621}$$

$$[T_N, \frac{\delta \log N}{\alpha} + t],\ \text{where } 0 < \delta < 1,\ t \in \mathbb{R}, \tag{3.622}$$

$$[\frac{\delta \log N}{\alpha}, \frac{\log N}{\alpha} + t],\ \text{where } 0 < \delta < 1, \quad t \in \mathbb{R}. \tag{3.623}$$

The first two time regimes are needed in proving the Propositions 1.9-1.11, the two remaining ones are necessary to prepare the stage for Subsection 3.4 where we shall relate ${}^*\mathcal{W}$ and $\mathcal{W}^*$.

Outline of Subsection 3.3

We now give a description of the evolution through the stages corresponding to the time intervals given in (3.620)-(3.623).

The first step is to examine the sparse set of sites at which the mutant population is of order $O(1)$ in a fixed finite time interval and then to describe this in terms of a Poisson approximation in the limit $N \to \infty$. This happens in Subsubsection 3.3.1 and proves Proposition 1.10.

In Subsubsection 3.3.2 we analyse the consequences for the longtime properties and prove Proposition 1.11 and in Subsubsection 3.3.3 the Proposition 1.9.

Subsequently in 3.3.4 we recall some related explicit calculations and in 3.3.5 and 3.3.6 we continue with the time intervals (3.621)-(3.623) to exhibit the law and properties of $\mathcal{W}^*$.

3.3.1 Mutant droplet formation at finite time horizon

Recall the definition of $(\beth_t^{N,m})_{t \geq 0}$ in (1.13). Furthermore we recall the abbreviation:

$$\widehat{x}_2^N(t) = \sum_{j=1}^N x_2^N(j,t) \tag{3.624}$$

for the total mass of type 2 in the whole population of N sites. We have to prove the convergence of $(\beth_t^{N,m})_{t \geq 0}$ as $N \to \infty$ to $(\beth_t^m)_{t \geq 0}$ and then we have to derive the properties of this limit as $t \to \infty$, which we had stated as three propositions which we now prove successively, but not in order, we close with Proposition 1.9.

Proof of Proposition 1.10

The strategy to prove the convergence in distribution of $(\daleth_t^{N,m})_{t\geq 0}$ as $N \to \infty$ is to proceed in steps as follows. We first consider only the contributions of mutation at a typical site where we have in reality two type of rare events, (1) the *immigration* of mass of type two from other sites and (2) the building up of this mass via *rare mutation* at this site. Indeed if we isolate the two effects and first suppress the immigration of type-2 mass, this simplification leads to N-independent sites. We therefore first look at one site, then at an independent collection of N sites and then finally build in the effect of immigration (destroying the independence) from other sites. In other words we work with the simplification in Step 1 and Step 2 and return in Step 3 to the original model. In Step 1 and Step 2 each the main point is condensed in a Lemma.

Step 1 *(Palm distribution of a single site dynamic)*

Recall Lemma 1.4 and the definition (1.36) of the single site excursion measure $\mathbb{Q}$ for the process without mutation. We must now consider the analogue for the process with mutation. Consider the *single site* dynamics (with only emigration but no immigration), which is given (misusing notation):

$$\begin{aligned} dx_2^N(1,t) &= -cx_2^N(1,t)dt + s\,x_1^N(1,t)x_2^N(1,t)dt + \frac{m}{N}(1-x_2^N(1,t))dt \\ &\quad + \sqrt{d\cdot(1-x_2^N(1,t))(x_2^N(1,t)}dw_2(i,t), \\ x_2^N(1,0) &= \frac{a}{N}. \end{aligned} \tag{3.625}$$

We use size-biasing to focus on the set of sites at which mutant mass appears in a finite time interval and use the excursion law of the process without mutation (with law P) to express the excursions of the process with mutation (with law $\widetilde{P}$). We prove first as key tool the following. Let

$$\mathbb{Q} \in \mathcal{M}(C(\mathbb{R}^+,[0,1])) \tag{3.626}$$

be defined as the excursion law of the process in (1.33) (which is(3.625) with $m=0$ which then does not depend on N). Furthermore let

$$\widetilde{P}^\varepsilon = \mathcal{L}[(x_2^\varepsilon(t))_{t\geq 0}] = \mathcal{L}[(x_2^\varepsilon(1,t))_{t\geq 0}] \text{ and } x_2^\varepsilon \text{ solves (3.625) with } x_2^\varepsilon(0) = \frac{a}{N} = 0, \frac{m}{N} = \varepsilon. \tag{3.627}$$

Furthermore let

$$\begin{aligned} W_{\text{ex}} := \quad & \{w \in C([0,\infty),\mathbb{R}^+),\ w(t) = 0 \text{ for } t < \zeta_b \text{ and } t > \zeta_d,\ w(t) > 0 \\ & \text{ on the interval } \zeta_b < t < \zeta_d \text{ for some } \zeta_b < \zeta_d \in (0,\infty)\}. \end{aligned} \tag{3.628}$$

Note that then ζ_b and ζ_d are unique functions of the element $w \in W_{\text{ex}}$. We shall use these functionals

$$\zeta_b, \zeta_d : W_{\text{ex}} \longrightarrow (0,\infty) \tag{3.629}$$

below.

Next observe that every continuous path starting at 0 and having an unbounded set of zeros we can write uniquely as a sum of elements of W_{ex} with disjoint intervals of positivity and with every point x not a zero of the path we can associate a unique excursion between

$$\zeta_b^x \text{ and } \zeta_d^x. \tag{3.630}$$

Lemma 3.19 *(Single site excursion law and Palm distribution)*

Consider the evolution of (3.625) and let $a = 0$, $d > 0$, $c \geq 0$, $t_0 > 0$. Then the following properties hold:

(a) Fix $t_0 > 0$ and $K \in \mathbb{N}$. For $k = 1, \cdots, K$ set $I_k := x_2(\frac{k}{K}t_0)$. Let $A_k(a_3)$ denote the event that the continuing trajectory satisfies, recall (3.628) and the sequel, $\zeta_d^x - \frac{k}{K} > a_3$ with $x = \frac{k}{K}t_0$. Furthermore consider the same quantities for processes $x_{2,k}$ and denote then by $\bar{I}_k$, respectively $\bar{A}_k$.

Replace in (3.625) the mutation term by εf_k and f_k is the indicator of $\left[\frac{k-1}{K}t_0, \frac{k}{K}t_0\right]$. These processes are called $x^\varepsilon_{2,k}$.

Then the $\bar{I}_k$ are identically distributed random variables with means $\frac{\varepsilon}{K}t_0$ and variance asymptotically of the form $\frac{const\cdot\varepsilon}{K}$ as $K \to \infty$.

Then the probability of more than one such excursion is

$$\lim_{\varepsilon\to 0} \widetilde{P}^\varepsilon\Big(\sum_{k=1}^K 1(\bar{A}_k) > 1 \Big| \sum_{k=1}^K 1(\bar{A}_k) \geq 1\Big) = 0. \tag{3.631}$$

This remains true for the I_k and A_k.

(b) The following family of measures $\widetilde{\mathbb{Q}}((a_1, a_2), a_3, \cdot)$ and a measure $\mathbb{Q}$ on $W_{\rm ex}$ exists. Choose $a_1, a_2 \in [0, \infty), a_1 < a_2, a_3 < 0$ and $A \in \mathcal{B}(C([0,\infty), \mathbb{R}^+))$. Then

$$\begin{aligned}\widetilde{\mathbb{Q}}((a_1,a_2),a_3,A)) &=: \widetilde{\mathbb{Q}}(\zeta_b(w) \in (a_1,a_2),\, \zeta_d - \zeta_b > a_3,\ w \in A)\\ &= \lim_{\varepsilon\to 0} \frac{\widetilde{P}^\varepsilon(\exists x \in (a_1,b_1)\zeta_b^x(w) \in (a_1,a_2),\, \zeta_d^x - \zeta_b > a_3,\ w \in A)}{\varepsilon}.\end{aligned} \tag{3.632}$$

The measures $\widetilde{\mathbb{Q}}$ on $W_{\rm ex}$ can be represented as ($\mathbb{Q}$ as in (3.626)),

$$\widetilde{\mathbb{Q}}((a_1,a_2),\, a_3,\, A) = \int_{a_1}^{a_2} \mathbb{Q}(\zeta > a_3, w(\cdot - s)) \in A)ds. \tag{3.633}$$

(c) Now consider (3.625) which has mutation rate $\varepsilon = \frac{m}{N}$ and put $a = 0$. Then for $t_0 > 0$

$$\lim_{N\to\infty} N \cdot E[x_2^N(1,t_0)] > 0. \tag{3.634}$$

(d) Let $\widehat{\mu}_t^N$ be as defined in (1.14), that is, the Palm distribution of $x_2^N(1,t)$ in (3.625) with $a = 0$. The following limit exists:

$$\widehat{\mu}_t^\infty(dx) = \lim_{N\to\infty} \widehat{\mu}_t^N(dx) = \frac{x\int_0^t \widetilde{\mathbb{Q}}(\zeta_b \in ds, \zeta_d > t, w(t) \in dx)ds}{\int\limits_0^\infty \int_0^t x\widetilde{\mathbb{Q}}(\zeta_b \in ds, \zeta_d > t, w(t) \in dx)ds}. \tag{3.635}$$

Hence the Palm distribution $\widehat{\mu}_t^N$ is the law of a random variable of order $O(1)$ which is also asymptotically non-degenerate as $N \to \infty$.

Furthermore as a consequence of (3.634) and (3.635) the first moment of the Palm distribution of $x_2^N(1,t_0)$ has mean satisfying:

$$\lim_{N\to\infty} \widehat{E}[x_2^N(1,t_0)] = \lim_{N\to\infty} \frac{E[(x_2^N(1,t_0))^2]}{E[(x_2^N(1,t_0))]} \in (0,\infty). \qquad \square \tag{3.636}$$

Proof of Lemma 3.19

We prove separately the parts a), b) and then c), d) of the Lemma.

(a) and (b).

To prove (a) and (b) we begin by approximating the single site process $\{x_2^\varepsilon(t) : 0 \leq t \leq t_0\}$ with law $\widetilde{P}^\varepsilon$ (which satisfies (3.625) with $\frac{m}{N} = \varepsilon$) by a sequence of processes, where we replace

the mutation term (induced by the rare mutation) in different time intervals by mutation terms at a grid of discrete time points getting finer and finer. In order to keep track of the mutations at different times we split the type 1 into different types. Namely let $K \in \mathbb{N}$ and consider a $K+1$ type Wright-Fisher diffusion with law

$$\widetilde{P}^{\varepsilon,K}, \tag{3.637}$$

which is starting with type 0 having fitness 0 and the other types having fitness $1 > 0$ (selection at rate s) and with mutation from type 0 to type k at rate ε only during the interval $(\frac{(k-1)t_0}{K}, \frac{kt_0}{K}]$.

We first consider a simpler process, which is built up from K-independent processes. Namely consider independent processes $x^{\varepsilon}_{2,k}$ representing the masses of types $k = 1,\dots,K$ in which in each time interval we suppress mutation of all types except some k. Then $x^{\varepsilon}_{2,k}(\cdot)$ is a solution of (3.625) with mutation term $\varepsilon 1_{(\frac{k-1}{K}t_0, \frac{k}{K}t_0]}(\cdot)$. Let $I_k(\varepsilon) := x^{\varepsilon}_{2,k}(1, \frac{k}{K}t_0)$. Let $A_k(a_3)$ denote the event that the continuing trajectory satisfies $\zeta_d - \frac{k}{K} > a_3$. Note that the $I_k(\varepsilon)$ are identically distributed random variables with means $\frac{\varepsilon}{K}t_0$ and variance asymptotically of the form $\frac{\varepsilon t_0^2}{2K^2} + o(\frac{\varepsilon}{K^2})$.

Note that as $\varepsilon \to 0$.

$$\frac{1}{\varepsilon} P_{I_k(\varepsilon)}(A_k(a_3)) = \frac{I_k(\varepsilon)}{\varepsilon}\mathbb{Q}(\zeta > a_3) + o(I_k(\varepsilon)). \tag{3.638}$$

Therefore

$$\begin{aligned} &\sum_{k=1}^{K} 1(\frac{kt_0}{K} \in (a_1,a_2))\frac{1}{\varepsilon} P_{I_k(\varepsilon)}(A_k(a_3)) \\ &= \frac{1}{K}\sum_{k=1}^{K} 1(\frac{kt_0}{K} \in (a_1,a_2))\frac{KI_k(\varepsilon)}{\varepsilon}\mathbb{Q}(\zeta > a_3) + o(\varepsilon) \text{ as } \varepsilon \to 0. \end{aligned} \tag{3.639}$$

Moreover by independence, $Prob(A_k \cap A_{k'}) = P_{I_k(\varepsilon)}(A_k)P_{I_{k'}(e)}(A_{k'}) = O(\varepsilon^2),\ k \neq k'$, and therefore the probability of more than one such excursion in the limit $\varepsilon \to 0$ is equal to zero.

$$\lim_{\varepsilon \to 0} \frac{1}{K^2}\sum_{k,k'} \frac{1}{\varepsilon} P_{I_k(\varepsilon)}(A_k) \cdot P_{I_{k'}(\varepsilon)}(A_{k'}) = 0. \tag{3.640}$$

Then letting $K \to \infty$ we get using the continuity of $t \to \mathbb{Q}(w(t) \in \cdot)$ that the following limit exists

$$\begin{aligned} &\widetilde{\mathbb{Q}}(\zeta_b \in (a_1,a_2), \zeta_d - \zeta_b > a_3, w(t_0) > x) \\ &= \lim_{\varepsilon \to 0}\lim_{K\to\infty}] \left[\frac{1}{K}\sum_k 1(\frac{kt_0}{K} \in (a_1,a_2))\frac{KI_k(\varepsilon)}{\varepsilon}\frac{1}{I_k(\varepsilon)} P_{I_k(\varepsilon)}(A_k \cap \{x(t_0) > x\})\right] \\ &= \lim_{\varepsilon\to 0}\left[\lim_{K\to\infty}\frac{1}{K}\sum_k 1(\frac{kt_0}{K} \in (a_1,a_2))\frac{KI_k(\varepsilon)}{\varepsilon}\mathbb{Q}(\zeta > a_3,\, w(t - \frac{kt_0}{K}) > x) + o(\varepsilon)\right] \\ &= \int_{a_1}^{a_2} \mathbb{Q}(\zeta > a_3,\, w(t_0 - s) > x)ds. \end{aligned} \tag{3.641}$$

We claim that the analogous result is valid for the multitype model, that is, letting A^*_k denote the presence of an excursion of type k of length greater that a_3 and having at time t_0 mass bigger than x, starting in (a_1,a_2)

$$\lim_{\varepsilon\to 0} \widetilde{P}^{\varepsilon,K}(\bigcup_{k=1}^{K} A^*_k) = \int_{a_1}^{a_2} \mathbb{Q}(\zeta > a_3, w(t_0 - s) > x)ds. \tag{3.642}$$

To verify that this remains true with dependence between the types as they arise in a multitype Fisher-Wright diffusion, we return to the law $\widetilde{P}^{\varepsilon,K}$. The result then follows from (3.641) by noting that

$$\widetilde{P}^{\varepsilon,K}(A_k \cap A_{k'}) \le P_{I_k(\varepsilon)}(A_k) \cdot P_{I_{k'}(\varepsilon)}(A_{k'}). \tag{3.643}$$

The latter holds, since

(c) and (d)

The proof of the convergence of $\widehat{\mu}_t^N$ as $N \to \infty$ proceeds by showing that for every $k \in \mathbb{N}$ we have:

$$\lim_{N\to\infty} N \cdot E([x_2^N(i,t)]^k) = m \lim_{\delta \downarrow 0} \int_0^1 x^k \int_0^t \widetilde{\mathbb{Q}}(\zeta_b \in ds, \zeta_d - \zeta_b > \delta, w(\cdot - s) \in dx) \tag{3.644}$$

and that the limiting variance is positive.

We first use the dual process to the process defined in equation (3.625) to compute the first and second moments. Since we discuss the limit $N \to \infty$ over a finite time horizon with finitely many initial particles, we get in the limit no immigration term, but only emigration at the migration rate.

We warn the reader at this point that we use the same notation for the dual process of this single site diffusion as for the one of our interacting system. Let $\Pi_u^{(j)}$ denote the number of factors in (3.7) at time t starting with j factors at time 0 where $\Pi_t^{(j)}$ is the birth and death process with birth and death rates

$$sk \text{ and } ck + \frac{d}{2}k(k-1) \text{ respectively.} \tag{3.645}$$

If we consider finite N we have additional immigration at rate $N^{-1}(\Pi_t^{(1),N} - k)^+$ if at the site considered we have k particles. This means that as $N \to \infty$ and over a finite time horizon with probability tending to 1 at rate $O(N^{-1})$ no immigration jumps occur.

Then for fixed t we have that:

$$E[x_2^N(i,t)] = 1 - E[x_1^N(i,t)], \tag{3.646}$$

$$\begin{aligned} E[x_1^N(i,t)] &= E[\exp(-\tfrac{m}{N}\textstyle\int_0^t \Pi_u^{(1),N} du)] \\ &\sim E[\exp(-\tfrac{m}{N}\textstyle\int_0^t \Pi_u^{(1)} du)], \quad \text{as } N \to \infty \\ &\sim 1 - \tfrac{m}{N} E[\textstyle\int_0^t \Pi_u^{(1)} du], \quad \text{as } N \to \infty \\ &\sim 1 - \tfrac{mc_1 t}{N}, \quad \text{as } N \to \infty, \end{aligned} \tag{3.647}$$

where $c_1 = c_1(t) \in (0,\infty)$ is constant in N. Hence

$$\lim_{N\to\infty} N \cdot E[x_2^N(i,t)] = mc_1 = m \int_0^t E[\Pi_u^{(1)}] ds. \tag{3.648}$$

For fixed t, as $N \to \infty$ we calculate the second moment:

$$\begin{aligned} E[(x_1^N(i,t))^2] &= E[\exp(-\tfrac{m}{N}\textstyle\int_0^t \Pi_u^{(2),N} du)] \\ &\sim E[\exp(-\tfrac{m}{N}\textstyle\int_0^t \Pi_u^{(2)} du)] \\ &\sim 1 - \tfrac{mc_2(t)}{N}, \end{aligned} \tag{3.649}$$

for some $c_2 = c_2(t) \in (0, \infty)$. Hence

$$E[x_2^N(i,t)^2] \begin{array}{l} = 1 - 2E[x_1^N(i,t)] + E[x_1(i,t)^2] \\ \sim 1 - 2E[\exp(-\frac{m}{N}\int_0^t \Pi_u^{(1)} du)] + E[\exp(-\frac{m}{N}\int_0^t \Pi_u^{(2)} du)] \\ \sim 2\frac{mc_1 t}{N} - \frac{mc_2 t}{N}, \quad \text{as } N \to \infty. \end{array} \tag{3.650}$$

Therefore

$$\lim_{N\to\infty} N \cdot E[x_2^N(i,t)^2] = 2mc_1(t) - mc_2(t) = m[2\int_0^t E[\Pi_u^{(1)}]du - \int_0^t E[\Pi_u^{(2)}]du]. \tag{3.651}$$

Therefore the l.h.s. of (3.636) equals

$$\frac{(2c_1(t) - c_2(t))}{c_1(t)}. \tag{3.652}$$

This limit is larger than 0 since (for every t)

$$c_2(t) < 2c_1(t) \tag{3.653}$$

holds because of the positive (uniformly in N) probability of coalescence between the two initial particles of the two populations before a migration step of either one. This implies that every limit point of the sequence of laws of $x_2^N(t)$ (solving (3.625)) has positive variance.

Similarly we introduce

$$c_k(t) = \int\limits_0^t \Pi_u^{(k)} du \tag{3.654}$$

and we can establish writing

$$E[(x_2^N(i,t))^k] = \sum_{j=0}^k \binom{k}{j} (-1)^j E[(x_1^N(i,t))]^j] \tag{3.655}$$

and then using

$$E[(x_1^N(i,t))^j] = E[exp(-\frac{m}{N}\int\limits_0^t \Pi_u^{(j)} du] \tag{3.656}$$

the convergence of all k-th moments as for $k = 1, 2$. This then gives the weak convergence of the Palm distribution $\widehat{\mu}_t^N$ as $N \to \infty$.

Finally we have to verify the relation (3.635), that is, the limiting Palm distribution is represented in terms of the excursion measure.

Note that for every $k \in \mathbb{N}$ we know that:

$$\begin{aligned} &\int_0^1 x^k \widetilde{\mathbb{Q}}_t(dx) \\ &= \int_0^t \int_0^1 x^k \mathbb{Q}(x_2(t-s) \in dx) ds = \int_0^t \int_0^1 x^k (\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \widetilde{P}^{\varepsilon}[x_2(t-s) \in dx) ds \\ &= \int_0^t \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E_\varepsilon[(x_2(t-s))^k] ds. \end{aligned} \tag{3.657}$$

Using the duality we calculate then (with $\widetilde{E}_\varepsilon$ denoting the expectation with respect to $\widetilde{P}_\varepsilon$)

$$(3.658)\qquad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\widetilde{E}_\varepsilon[x_2(u)] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E[1-(1-\varepsilon)^{\Pi_u^{(1)}}] = E[\Pi_u^{(1)})].$$

$$(3.659)\qquad \begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\widetilde{E}_\varepsilon[x_2(u)^2] &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(1-2\widetilde{E}_\varepsilon[x_1(u)]+\widetilde{E}_\varepsilon[x_1(u)^2]\Big)\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(1-2E[1-(1-\varepsilon)^{\Pi_u^{(1)}}]+E[1-(1-\varepsilon)^{\Pi_u^{(2)}}]\Big)\\ &= 2E[\Pi_u^{(1)}]-E[\Pi_u^{(2)}].\end{aligned}$$

Substituting this (with $k=2$) in (3.657) and comparing with (3.648) and (3.649) we verify (3.644) for $k=1,2$ and (3.636). Similar calculations can be used to verify the claim for all $k\in\mathbb{N}$. ■

Step 2 *(Compound Poisson limit)*

Consider a collection of processes where each component of the system is still as in (3.625) in which we ignore the effects of immigration into sites and only *emigration* is still accounted for, namely,

$$(3.660)\qquad \begin{aligned} dx_2^N(i,t) = -cx_2^N(i,t)dt \quad &+ \quad s\,x_1^N(i,t)x_2^N(i,t)dt + \frac{m}{N}(1-x_2^N(i,t))dt\\ &+ \quad \sqrt{d\cdot(1-x_2^N(i,t))x_2^N(i,t)}\,dw_2(i,t),\\ &\qquad x_2^N(i,0)=\frac{a}{N},\quad i=1,2,\cdots,N.\end{aligned}$$

Note that this is a system of N independent diffusion processes. Furthermore we calculate for the total mass process

$$(3.661)\qquad d\widehat{x}_2^N(t) \le ((-c+s)\widehat{x}_2^N(t)+\frac{m}{N})dt + dM_t,$$

with a martingale $(M_t)_{t\ge 0}$, where

$$(3.662)\qquad \langle M\rangle_t \le d\cdot\widehat{x}_2^N(t).$$

Therefore we can bound by a submartingale and if we start with only type 1, i.e. *if* $a=0$, then by submartingale inequalities we get:

$$(3.663)\qquad P[\sup_{t\le T}|x_2^N(i,t)|>\varepsilon] \le \frac{\text{constant}}{N}.$$

We return now to the study of the independent collection of diffusions we introduced in the beginning of this step. Consider the atomic measure-valued process

$$(3.664)\qquad (\widetilde{\beth}_t^{N,m})_{t\ge 0},$$

which is defined as the analogue of $(\beth_t^{N,m})_{t\ge 0}$ but with only excursions arising from mutation, that is ignoring immigration at each site (emigration is still accounted for) which was defined in (3.654). Similarly let

$$(3.665)\qquad \{\widetilde{x}_2^N(i,t), i\in\mathbb{N}\}$$

be the corresponding collection of independent diffusions. Then we carry out the limit $N\to\infty$.

Lemma 3.20 *(Droplet growth in absence of immigration)*

At time t_0, asymptotically as $N \to \infty$ we have the following three properties for the dynamics introduced above in (3.664, 3.665).

(a) There is a Poisson number of sites j at which $\widetilde{x}_2^N(j,t_0) > \varepsilon$ where the parameter is

$$m \int_0^{t_0} \mathbb{Q}(w(t_0 - s) > \varepsilon) ds. \tag{3.666}$$

(b) At time t_0, in the limit $N \to \infty$, there are a countable number of sites contributing to a total mutant mass of order $O(1)$.

(c) For each $t > 0$ the states of $\widetilde{\beth}_t^{N,m}$ converge in the sense of the weak atomic topology to

$$\widetilde{\beth}_t^m \text{ on } [0,1], \tag{3.667}$$

with atomic random measure specified by the two requirements

$$\text{atom locations i.i.d. uniform on } [0,1] \tag{3.668}$$

and atom masses given by a Poisson random measure on $[0,1]$ with intensity measure

$$m\nu_t(dx), \text{ where } \nu_t(dx) = \int_0^t \mathbb{Q}(w : w(t-s) \in dx) ds. \tag{3.669}$$

The Palm measure $\widehat{\nu}_t$ of ν_t satisfies (recall $\widehat{\mu}_t^\infty$ from Lemma 3.19 (d)):

$$\widehat{\nu}_t = \widehat{\mu}_t^\infty. \quad \square \tag{3.670}$$

Proof of Lemma 3.20 (a) It follows from Lemma 3.19 (a) that

$$\lim_{N\to\infty} N \cdot P[x_2^N(i,t) > \varepsilon] = m\widetilde{\mathbb{Q}}_t((\varepsilon, 1]). \tag{3.671}$$

Therefore in the collection of N sites it follows that asymptotically as $N \to \infty$ the number of sites j with $\widetilde{x}_2^N(j,t_0) > \varepsilon$ (recall that in this step we suppress immigration and we are considering only mass originating at this site) is Poisson with parameter $m\widetilde{\mathbb{Q}}_t((\varepsilon, 1])$.

(b) Since $\nu_t((0,\infty)) = \infty$ and $\nu_t((\delta,\infty)) < \infty$ for $\delta > 0$, there are countably many atoms in the limit as $N \to \infty$.

(c) Consider the sequence obtained by size-ordering the atom sizes in $\widetilde{\beth}_t^{N,m}$. Note that the limiting point process on $[\varepsilon, 1]$ is given by a Poisson number with i.i.d. sizes and the distribution of the atoms is given by $\frac{\widetilde{\mathbb{Q}}_t(dx)}{\widetilde{\mathbb{Q}}_t([\varepsilon,1])}$. It follows that the order statistics also converge and therefore the joint distribution of the largest k atoms converge as $N \to \infty$.

To verify that the limit is pure atomic we note that the expected mass of the union of sites of size smaller than ε converges to 0 as $\varepsilon \to 0$ uniformly in N by the explicit calculations in the next section (cf. (3.706)). The convergence of $\widetilde{\beth}_t^{N,m}$ in the weak atomic topology then follows by Lemma 1.6.

Step 3 *(Completion of proof of convergence)*

Here we have to incorporate the migration of mass between sites in particular the *immigration* to a site from all the other sites. Note that the immigration rate at fixed site is as the rare mutation of order N^{-1} as long as the total mass is $O(1)$ as $N \to \infty$ and therefore we get here indeed an effect comparable to the one we found in Step 1 and 2. We can show directly that the first two moments of the total mass $\widehat{x}_2^N(t) = \beth_t^{N,m}([0,1])$ converge to the first two moments of $\widehat{x}_2(t) = \beth^m(t)([0,1])$. These moment calculations are carried out later on in Subsection 3.5, in particular the first moment result is given by (3.861) and the second moment is given by (3.918). This gives the tightness of the marginal distributions.

Consider the sequence of time grids $\{\frac{\ell}{2^k},\ \ell = 0,1,2,\dots\}$ with index k and with width 2^{-k}. Then the migration of mass between time $\frac{\ell}{2^k}$ and $\frac{\ell+1}{2^k}$ and the mutation in this interval will be replaced by immigration from other sites and by rare mutation at the *end* of the interval. We then define using this idea an approximate evolution in discrete time by an approximate recursion scheme of atomic measures on $[0,1]$.

Let $T_t(x)$ denote the evolution of atomic measures, where each atom follows the single site Fisher-Wright diffusion with emigration and the initial atomic measure is x. Furthermore let $Y_\ell^{N,k}$ be an atomic measure with a countable set of new atoms such that the atoms in $Y_\ell^{N,k}$ are produced as above in Step 1 if we observe the process at time 2^{-k} but now the intensity of production of a new atom is instead of just m in the previous steps now given by:

$$(3.672) \quad m + c\widehat{x}_2^N(\frac{\ell}{2^k}),$$

where $\widehat{x}_2^N$ is given in (3.624). Then define:

$$(3.673) \quad \widetilde{X}_2^N(\frac{\ell+1}{2^k}) = T_{\frac{1}{2^k}}\widetilde{X}_2^N(\frac{\ell}{2^k}) + Y_\ell^{N,k}, \quad k \in \mathbb{N},$$

which defines, for each N and a fixed parameter $k \in \mathbb{N}$, a piecewise constant atomic measure-valued process which we denote by

$$(3.674) \quad (\daleth_t^{N,k,m})_{t\geq 0}.$$

We will now first investigate the weak convergence of this process, in the parameters k, N tending to infinity. First we focus on the weak convergence w.r.t. the topology of the weak convergence of finite measures on the state space and then we pass to the (stronger) one w.r.t. the weak atomic topology on the state space.

Consider first $N \to \infty$. For given k and t, as $N \to \infty$, the random variable $\daleth_t^{N,k,m}$, converges weakly even in $\mathcal{M}_a([0,1])$ to $\daleth_t^{\infty,k,m}$ as proven in Lemma 3.20 (c). We can then obtain the convergence of the finite dimensional distributions of $(\daleth_t^{N,k,m})_{t\geq 0}$ to those of $(\daleth_t^{\infty,k,m})_{t\geq 0}$. Tightness and convergence is obtained as we shall argue at the end of this proof so that we have as $N \to \infty$ weak convergence of processes to

$$(3.675) \quad (\daleth_t^{\infty,k,m})_{t\geq 0}.$$

Next we let $k \to \infty$ and consider the weak topology on the state space. Note first that the random variable in (3.675) is by construction *stochastically increasing in* k, since passing from k to $k+1$ we pick up additional contributions from newly formed atoms. We can then show that as $k \to \infty$ the $\daleth_t^{\infty,k,m}$ converge in distribution to $\daleth_t^m$. Similarly we can take the limit $k \to \infty$ in $\daleth^{N,k,m}$ to get $\daleth^{N,m}$.

We can then use the triangle inequality for the Prohorov metric $\rho(\daleth_t^{N,m}, \daleth_t^m)$ to get our result. Namely from the weak convergence of $\daleth_t^{N,k,m}$ to $\daleth_t^{\infty,k,m}$ as $N \to \infty$, the convergence as $k \to \infty$ of $\daleth_t^{N,k,m}$ to $\daleth_t^{N,m}$ as well as the convergence as $k \to \infty$ of $\daleth_t^{\infty,k,m}$ to $\daleth_t^m$. Using then the Markov property and a standard argument we can verify the convergence of the finite dimensional distributions of $\daleth_t^{N,m}$ to the finite dimensional distributions of $\daleth_t^m$.

It now remains to check for $(\daleth^{N,m})_{t\geq 0}$ as $N \to \infty$ the tightness condition of the laws and then convergence actually holds also in $C([0,\infty), (\mathcal{M}_a([0,1]), \rho_a))$. We first prove tightness with respect to the *weak topology* on the state space, i.e. we first prove tightness in the path space $C([0,\infty), \mathcal{M}_f([0,1]))$.

We write the state of the process as

$$(3.676) \quad \daleth_t^{N,m} = \sum_{i=1}^N a_i^N(t)\delta_{x_i^N(t)}.$$

Then it suffices to show that the laws of

$$(3.677)\quad \{\sum_{i=1}^{N} f(a_i^N(t), x_i^N(t)), \quad N \in \mathbb{N}\}$$

are tight for a bounded continuous function f on $[0,1] \times [0,1]$. But the $x_i^N(t)$ do not change in time and the $a_i^N(t)$ are semimartingales with bounded characteristics and are therefore tight by the Joffe-Métivier criterion.

To get to the convergence in path space based on the *weak atomic topology* on the state space of the process, we next note that since the joint distributions of the ordered atom sizes at fixed times converge, the locations are constant in time (as long as they are charged), we have convergence of the finite dimensional distributions in the weak atomic topology. The verification of the condition for tightness in $C([0,\infty), \mathcal{M}_a([0,1]))$ (1.63) follows as in the proof of Theorem 3.2 in [EK4].

3.3.2 The long-term behaviour of limiting droplet dynamics (Proof of Proposition 1.11)

We now investigate the behaviour of the limiting ($N \to \infty$) droplet dynamic, where again the dual process is the key tool, now the dual process of McKean-Vlasov dynamics and certain subcritical Fisher-Wright diffusions.

Proof of Proposition 1.11

We prove separately the three parts of the proposition.

a) Let $(\daleth_t^m)_{t\geq 0}$ be defined as in Propositions 1.5. We begin by deriving a formula for the first moment

$$(3.678)\quad m(t) = E[\daleth_t^m([0,1])],$$

given $\daleth_0^m(\cdot)$ with $\daleth_0^m([0,1]) < \infty$, which is accessible to an asymptotic analysis.

First we introduce a key ingredient in the formula for $m(\cdot)$ and we obtain its relation to the exponential growth rates α^* respectively α.

Let

$$(3.679)\quad f(t) = \int_{W_0} w(t)\mathbb{Q}(dw) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} E_\varepsilon[\widehat{x}_2(t)],$$

where $\widehat{x}_2(\cdot)$ denote the solution of (1.33) and E_ε refer to the initial state ε. We now obtain an expression for (3.679).

To compute the first moment of $\widehat{x}_2(t)$ we use the dual representation for the moments of the SDE (1.33). The dual process to be used then is $(\eta_t, \mathcal{F}_t^+)_{t\geq 0}$ with one initial factor $1_1(\cdot)$. The dual particle process $(\eta_t)_{t\geq 0}$ then is effectively a birth and death process denoted $(\widetilde{D}_0(t))_{t\geq 0}$ on $\mathbb{N}_0$, with a dynamic given by the evolution rule that the process can jump up or down by 1 (birth or death) and the rates in state k are given by:

$$(3.680)\quad \text{birth rate } sk, \text{ death rate } ck + \frac{d}{2}k(k-1)$$

and

$$(3.681)\quad \text{initial state } \widetilde{D}_0 = 1.$$

Then the dual expression is given by

$$(3.682)\quad E_\varepsilon[\widehat{x}_2(t)] = 1 - E[(1-\varepsilon)^{\widetilde{D}_0(t)}].$$

Therefore

$$(3.683)\quad f(t)=\int_{W_0} w(t)\mathbb{Q}(dw)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E_\varepsilon[\widehat{x}_2(t)]=\lim_{\varepsilon\to 0}\frac{E[1-(1-\varepsilon)^{\widetilde{D}_0(t)}]}{\varepsilon}=E[\widetilde{D}_0(t)].$$

Now let α^* be chosen so that

$$(3.684)\quad c\int_0^\infty e^{-\alpha^* r}f(r)dr=c\int_0^\infty e^{-\alpha^* r}E[\widetilde{D}_0(r)]dr=1.$$

Recalling that by (3.138), (recall the death rate of $D_0(t)$ is zero if $k-1$)

$$(3.685)\quad \int_0^\infty e^{-\alpha^* r}E[\widetilde{D}_0(r)]dr=\int_0^\infty e^{-\alpha^* r}E[D_0(r)1_{D_0(r)\geq 2}]dr,$$

it follows that $\alpha^*=\alpha$ with α as defined in (3.135), (3.137).

Now we are ready to write down the final equation for $m(\cdot)$. Consider

$$(3.686)\quad \beth^*_0(t)=(\sum y_i(t)\delta_{a_i})_{t\geq 0},$$

where $y_i(t)$, $\sum y_i(0)<\infty$, are those realizations of the (independent) Fisher-Wright diffusions (1.33) which represent the atoms that were present at time 0 without immigration. Furthermore assume that $\sum\limits_i y_i(0)<\infty$. Then we consider the process $(\beth^m_t)_{t\geq 0}$ defined in Proposition 1.5 with $\beth^*$ therein as given in (3.686).

By taking expectations in the stochastic equation defining $\beth^m$, namely (1.46), we get if we define

$$(3.687)\quad g(t)=E[\sum_i y_i(t)],$$

the renewal equation

$$(3.688)\quad \begin{aligned} m(t)=E[\beth^m_t([0,1])] &= E[\sum_i y_i(t)]+\textstyle\int_0^t E[(m+c\beth^m_r([0,1]))\int_{W_0}w(t-r)\mathbb{Q}(dw)]dr\\ &= g(t)+m\textstyle\int_0^t f(r)dr+\int_0^t cm(r)f(t-r)dr.\end{aligned}$$

The last step is now to analyse the growth behaviour of $(m(t))_{t\geq 0}$ using renewal theory. Multiplying through equation (3.688) by $e^{-\alpha^* t}$ we get an equation for $(e^{-\alpha^* t}m(t))_{t\geq 0}$ in terms of $(e^{-\alpha^* t}f(t))$ and $(e^{-\alpha^* t}g(t))$ as follows:

$$(3.689)\quad e^{-\alpha^* t}m(t)=e^{-\alpha^* t}[(g(t)+m\int_0^t f(r)dr)]+c\int_0^t(e^{-\alpha^* r}m(r))(e^{-\alpha^*(t-r)}f(t-r))dr.$$

This equation in $e^{-\alpha^* t}f(t)$ and $e^{-\alpha^* t}g(t)$ has the form of a renewal equation.

We want to apply now a *renewal theorem* to this equation. Define R by

$$(3.690)\quad R=c\int_0^\infty re^{-\alpha^* r}f(r)dr.$$

Since $f(t)$ and $g(t)$ are continuous and converge to 0 exponentially fast, it can be verified that $a(u)=e^{-\alpha^* u}\left[g(u)+m\int_0^u f(r)dr\right]$ is directly Riemann integrable. Therefore by the renewal theorem ([KT], Theorem 5.1) we obtain from (3.689) that:

$$(3.691)\quad \lim_{t\to\infty}e^{-\alpha^* t}m(t)=\frac{1}{R}\int_0^\infty e^{-\alpha^* u}\left[g(u)+m\int_0^u f(r)dr\right]du\in(0,\infty).$$

Hence $e^{-\alpha^* t} E[\gimel^m(t)]$ converges as $t \to \infty$ to the r.h.s. of (3.691) which concludes the proof of part (a) of the proposition.

(b) Let

$$\widehat{x}_2(t) = \gimel_t^m([0,1]). \tag{3.692}$$

The convergence in distribution of $e^{-\alpha t}\widehat{x}_2(t)$ as $t \to \infty$ follows since by a) the laws of $\{e^{-\alpha t}\widehat{x}_2(t), \quad t \geq 0\}$ form a tight family and we have to exclude only that several limit points exist. We know that for any $\tau > 0$

$$\lim_{t\to\infty} E([e^{-\alpha(t+\tau)}\widehat{x}_2(t+\tau) - e^{-\alpha t}\widehat{x}_2(t)]^2) = 0, \tag{3.693}$$

which is proved in Proposition 3.27. (Recall here that over a finite time horizon $[0, \tau]$ the mutation can be ignored and hence the proposition applies). We have to strengthen this statement (3.693) to

$$\lim_{t\to\infty} (\sup_{\tau>0} E[e^{-\alpha(t+\tau)}\widehat{x}_2(t+\tau) - e^{-\alpha t}\widehat{x}_2(t))^2]) = 0. \tag{3.694}$$

This extension is provided in Corollary 3.28 of Subsubsection 3.3.6 where it is proved using moment calculations. This implies that there exists $\mathcal{W}^*$ such that

$$\widehat{x}_2(t) \longrightarrow \mathcal{W}^* \text{ in } L^2. \tag{3.695}$$

The non-degeneracy of the limit $\mathcal{W}^*$ follows from (c).

(c) This claim on the variance of $\mathcal{W}^*$ follows from Proposition 3.21 in Subsubsection 3.3.5.

Remark 41 *The proof of (c) demonstrates that the randomness arises from the early rare mutation events.*

3.3.3 Proof of Proposition 1.9

This proof builds mainly on the previous subsubsection and on calculations we shall carry out in Subsection 3.4.

(a) This follows from Proposition 1.10

(b) Recall that we know already that $e^{-\alpha t}\widehat{x}_2^N(t)$ converges in law as $N \to \infty$ to a limit $e^{-\alpha t}\widehat{x}_2(t)$ and hence by Skorohod embedding and our moment bounds we have convergence on some L^2 space, furthermore $e^{-\alpha t}\widehat{x}_2(t)$, converges in L^2 as $t \to \infty$ to a limit $\mathcal{W}^*$ as proved in (3.695) above. Here we shall show that

$$\lim_{t\to\infty} \lim_{N\to\infty} E[(e^{-\alpha t}\widehat{x}_2^N(t) - e^{-\alpha t_N}\widehat{x}_2^N(t_N))^2] = 0 \tag{3.696}$$

and then we use (1.79) to get the claim. The above relation follows via Corollary 3.31 which is proven by moment calculation which requires subtle coupling arguments which we shall develop in Subsubsections 3.3.5 and 3.3.6.

3.3.4 Some explicit calculations

The entrance law from 0 for the type-2 mass can also be derived using explicit formulas for densities of the involved diffusions both for Fisher-Wright and branching models. We collect this in the following two remarks.

Remark 42 *Consider the Fisher-Wright given by the SDE*

$$(3.697)\qquad \begin{aligned} dx(t) &= \frac{m}{N}[1-x(t)] + c[\frac{y(t)}{N} - x(t)]dt + sx(t)(1-x(t))dt + \sqrt{d \cdot x(t)(1-x(t))}dw(t),\\ x(0) &= p, \end{aligned}$$

where $y(t) = \widehat{x}_2^N(t)$ is inserted as an external signal. By Kimura [Kim1] if $m = c = s = 0$, then there is no N-dependence and the density of the solution at time t is

$$(3.698)\qquad \begin{aligned} &f(p,x;t)\\ &\quad= \textstyle\sum_{i=1}^{\infty} p(1-p)i(i+1)(2i+1)F(1-i,i+2,2,p)\cdot F(1-i,i+2,2,x)e^{i(i+1)dt/2}\\ &\quad= 6p(1-p)e^{-dt} + 30p(1-p)(1-2p)(1-2x)e^{-3dt} + \dots, \end{aligned}$$

where $F(\alpha,\beta,\gamma;x)$ is the hypergeometric function.

Consider now the neutral Fisher-Wright diffusion, $s = 0$ and constant $y(t) = m$, $c > 0$, $d = 1$. Crow-Kimura (see [GRD], Table 3.4, or Kimura [Kim1], (6.2)) obtained the density at time t

$$(3.699)\qquad \begin{aligned} \widetilde{f}^N(y,x,t) = \;& x^{\mu-1}(1-x)^{\nu-\mu-1}\\ &\times \textstyle\sum_{i=0}^{\infty} \frac{(\nu+2i-1)\Gamma(\nu+i-1)\Gamma(\nu-\mu+i)}{i!\Gamma^2(\nu-\mu)\Gamma(\mu+i)} \times \widetilde{F}_i(1-x)\widetilde{F}_i(1-y)\exp(-\lambda_i t), \end{aligned}$$

where the initial value of the process is y and

$$(3.700)\qquad \mu = \frac{2m}{N}, \nu = 2(\frac{m}{N}+c), \lambda_i = i[2(\frac{m}{N}+c)+(i-1)]/2,$$

$$(3.701)\qquad \widetilde{F}_i(x) = F(\nu+i-1,-i;\nu-\mu;x),$$

where F is the socalled hyper-geometric function that is that solution of the equation

$$(3.702)\qquad x(1-x)F'' + [\gamma - (\alpha+\beta+1)x]F'' - \alpha\beta F = 0,$$

which is finite at $x = 0$.

Setting $y = 0$ and taking $N \to \infty$, then for every $t > 0$ and $x \in [0,1]$, taking $\widetilde{f}^N(x,t) = \widetilde{f}^N(0,x,t)$, we get that there exists a function $\widetilde{f}^\infty(x,t)$ which is for every t the density of a σ-finite measure such that:

$$(3.703)\qquad N\widetilde{f}^N(x,t) \longrightarrow \widetilde{f}^\infty(x,t).$$

The limit defines an entrance type law. Note that $\widetilde{f}^N$ has the same behaviour near zero as $N \to \infty$ as the branching density below in (3.705).

Consider now the case $s > 0$. Then selection term then leads to a new law which is absolutely continuous w.r.t. the neutral one and the density is given by the Girsanov factor

$$(3.704)\qquad L_t = \exp\left[\frac{s}{d}\int_0^t \sqrt{d\cdot x(s)(1-x(s))}dw(t) - \frac{s^2}{2d^2}\int_0^t d\cdot x(s)(1-x(s))ds\right].$$

Therefore the selection term does not change the asymptotics since it involves a bounded Girsanov density (uniform in N).

Alternatively to the above argument we can take the Kimura solution (4.10) of the forward equation and take the limit $\frac{\phi(p,x;t)}{p}$ as $p \to 0$. $\square$

Remark 43 *For comparison, consider the branching approximation where there is no competition between excursions and we obtain for initial state 0 the Gamma density*

$$(3.705)\qquad f^N(x,t) = \frac{c}{N}x^{\frac{m}{N}-1}e^{-x/e^{\widetilde{s}t}},\ x \geq 0.$$

Then Nf^N converges to the density $x^{-1}e^{-x/e^{\widetilde{s}t}}$ on $\mathbb{R}^+$ of a σ-finite measure. Moreover

$$(3.706)\qquad N\cdot\int_0^\varepsilon xf^N(x,t)dx \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly in } N. \qquad \square$$

3.3.5 First and second moments of the droplet growth constant $\mathcal{W}^*$

Continuing the study of the droplet growth we now turn to times in $[T_0, T_N]$ as given in (3.621) and (3.622). In the sequel we compute the limits as $N \to \infty$ of the first two moments of the total type-2 mass in the population denoted $\widehat{x}_2^N(t_N)$, for $t_N \to \infty$, $\quad t_N = o(\log N)$. The present calculation allows in particular to show the existence of $\mathcal{W}^*$ and to determine its moments.

As before we use the dual representation to compute moments. For example, recall that

$$(3.707) \quad E[x_1^N(i, t_N)] = E[\exp(-\frac{m}{N} \int\limits_0^{t_N} \Pi_s^{N,1} ds)],$$

where $\Pi_t^{N,1}$ is of the total number of particles in the dual cloud in $\{1, \ldots, N\}$ starting with one factor $1_{\{1\}}$ at site i at time $t = 0$. As developed earlier this cloud is described by a CMJ process and has the form

$$(3.708) \quad \Pi_t^{N,1} = \sum_{j=1}^{W_N(t)e^{\alpha t}} \zeta_j^N(t),$$

where $W_N(t) \to W(t)$ as $N \to \infty$ and $W(t) \to W$ a.s. as $t \to \infty$ and the processes at the different sites satisfy $\mathcal{L}[(\zeta_j^N(t))_{t \geq 0}] \Longrightarrow \mathcal{L}[(\zeta_j(t))_{t \geq 0}]$ as $N \to \infty$, where the $(\zeta_j(t))_{t \geq 0}$ for $j \in \mathbb{N}$ are all evolving independently and are given by the birth and quadratic death process at an occupied site with the same dynamic but starting from one particle at the random time, when the site is first occupied. The dependence between the different occupied sites arises through these random times even in the $N \to \infty$ limits.

If we consider second moments we have to start with two particles and then get the analog of (3.707) but in the representation of the r.h.s. of this formula given in (3.708), $W_N(t)$ and ζ^N have to be replaced accordingly and in the limit $N \to \infty$ the $W = W^{1,1}$ is replaced by $W^{1,2}$.

We have noted above that if $t_N = o(\log N)$ then the probability that a collision between dual particles due to a migration step occurs up to time t_N goes to 0 as $N \to \infty$. *The new element that arises in this subsubsection is the fact that in the computation of the higher moments of* $\widehat{x}_2^N(t_N)$, *these collision events must be taken into account.* The reason for this is that we will compute moments of the mutant mass $\widehat{x}_2^N(t)$ in terms of the moments of $\widehat{x}_1^N(t)$. We see below from the formula (3.716) that in the expansion of k-th moments of the total mass powers of N appear, which forces us to consider in the expansions of these moments as well as higher order terms in N^{-1}. Hence we have to analyse more carefully the overlaps in the dual clouds corresponding to different ancestors which have probabilities which go to zero but not fast enough to be discarded.

The key tool we work with is the coloured particle systems (WRGB) introduced in Subsubsection 3.2.9.

The main result of whose proof this subsubsection is devoted to is:

Proposition 3.21 *(Limiting growth constant of droplet)*
Let $t_N \to \infty$ *and*

$$(3.709) \quad \limsup_{N \to \infty} \frac{t_N}{\log N} < \frac{1}{2\alpha}.$$

(a) The family

$$(3.710) \quad \{e^{-\alpha t_N} \widehat{x}_2^N(t_N) : N \in \mathbb{N}\} \text{ is tight,}$$

with non-degenerate limit points $\mathcal{W}^*$ with

$$0 < E[\mathcal{W}^*] < \infty, \quad 0 < Var[\mathcal{W}^*] < \infty. \tag{3.711}$$

(b) Furthermore:

$$\lim_{N\to\infty} e^{-\alpha t_N} E[\widehat{x}_2^N(t_N))] = m^* E[W_1], \tag{3.712}$$

$$\lim_{N\to\infty} e^{-2\alpha t_N} E[\widehat{x}_2^N(t_N))^2] = (m^*)^2(EW_1)^2 + 2m^*(EW_1)^2\kappa_2, \tag{3.713}$$

where α is defined as in (3.139), $\kappa_2 > 0$ (which is given precisely in (3.745)) and

$$m^* = \frac{m(\alpha+\gamma)}{c}, \text{ with } b = 1 + \frac{\gamma}{\alpha}. \tag{3.714}$$

(c) The third moment satisfies

$$\limsup_{N\to\infty} E[e^{-3\alpha t_N}(\widehat{x}_2^N(t_N))^3] < \infty, \tag{3.715}$$

so that the first and second moments of $\mathcal{W}^*$ are given by the formulas on the r.h.s. of (3.712) and (3.713). $\square$

The proof of the main result is given in the rest of this subsubsection which has five parts, namely we require four tools to finally carry out the proof.

(1) First we give a preparatory lemma 3.22 that gives asymptotic expressions for the moments of $\widehat{x}_2^N(t_N)$ in terms of the moments of $\widehat{x}_1^N(t_N)$.

(2) In order to calculate the moments of $\widehat{x}_1^N(t_N)$ we must determine the growth of dual clouds starting with one or two initial factors. This is stated in Proposition 3.24 and is then proved.

(3) Here we introduce a multicolour system to analyse the two dual clouds and their interaction starting both from one particle (at different sites).

(4) Then we present in Lemma 3.26 the tools to obtain the asymptotics of the expectations of the exponentials involving the occupation times of the dual clouds.

(5) Finally we complete the proof of the above main result, the Proposition 3.21, using the dual expressions at the end of the section and the results from the previous Parts 1-4.

Part 1 *(Moment formulas)*

Lemma 3.22 *(k-th moment formulas)*

(a) We have the k-th moment formula for the total type-2 mass:

$$\begin{aligned} \bar{m}_k^{N,2}(t_N) \quad &:= E[(\widehat{x}_2^N(t_N))^k] = E\left[\left(\sum_{i=1}^N x_2^N(i,t_N)\right)^k\right] \\ &= \sum_{\ell=1}^k \begin{pmatrix} k \\ \ell \end{pmatrix} (-1)^\ell N^{k-\ell}\left(\bar{m}_\ell^{N,1}(t_N)\right), \quad k = 1,2,\cdots \end{aligned} \tag{3.716}$$

where $\bar{m}_\ell^{N,1}$ is the ℓ-th moment for the total type-1 mass.

(b) The term on the r.h.s. of (3.716) is calculated as follows. Let $q = (q_1,\cdots,q_j)$ and let $\Pi^{N,q}$ be the dual starting with q_i-particles at site $i = 1,\cdots,j$. Define

$$\mu_{q_1,\ldots,q_j;q}(t_N) = E\left[\exp\left(-\frac{m}{N}\int_0^{t_N} \Pi_s^{N,(q_1,\cdots,q_j)} ds\right)\right]. \tag{3.717}$$

Then

$$\bar{m}_q^{N,1}(t_N) = \sum_{j=1}^q \begin{pmatrix} N \\ j \end{pmatrix} \sum_{q_1+\ldots q_j = q} \frac{q!}{q_1!\ldots q_j!}\mu_{q_1,\ldots,q_j;\,q}(t_N). \quad \square \tag{3.718}$$

Proof of Lemma 3.22

(a) First recall that

$$\widehat{x}_1^N(t) = \sum_{i=1}^N x_1^N(i,t), \quad \widehat{x}_2^N(t) = \sum_{i=1}^N x_2^N(i,t) = N - \widehat{x}_1^N(t) \tag{3.719}$$

and therefore

$$\begin{aligned} \bar{m}_k^{N,2}(t_N) \quad &:= E[(\widehat{x}_2^N(t_N))^k] = E\left[\left(\sum_{i=1}^N x_2^N(i,t_N)\right)^k\right] \\ &= \sum_{\ell=1}^k \begin{pmatrix} k \\ \ell \end{pmatrix} (-1)^\ell N^{k-\ell} E\left[\left(\sum_{i=1}^N x_1^N(i,t_N)\right)^\ell\right] \quad , \quad k = 1,2,\cdots \end{aligned} \tag{3.720}$$

Recalling the definition of $\bar{m}_\ell^{N,1}(t)$ the claim follows.

(b) Moreover we can express the moments of the mass of type 1 appearing on the r.h.s. of (3.720) by expanding according to the *multiplicity* of the occupation of sites in the mixed moment expression, namely:

$$\bar{m}_k^{N,1}(t_N) = E\left[\left(\sum_{i=1}^N x_1^N(i,t)\right)^k\right] \tag{3.721}$$

$$= \sum_{i_1 \neq i_1 \neq \ldots \neq i_k} E\left[x_1^N(i_1,t)\ldots x_1^N(i_k,t)\right] + \ldots + \sum_{i=1}^N E[(x_1^N(i,t))^k].$$

The moments on the r.h.s. above can be computed as follows. Define

$$\mu_{q_1,\ldots,q_j;\,q}(t_N) = E\left[\prod_{\ell=1}^j (x_1^N(i_\ell,t_N))^{q_\ell}\right], \quad i_\ell \text{ distinct}, \quad \sum_{\ell=1}^j q_\ell = q. \tag{3.722}$$

Then

$$\bar{m}_q^{N,1}(t_N) = \sum_{j=1}^q \begin{pmatrix} N \\ j \end{pmatrix} \sum_{q_1+\ldots q_j = q} \frac{q!}{q_1!\ldots q_j!} \mu_{q_1,\ldots,q_j;\,q}(t_N). \tag{3.723}$$

Next use the duality relation to express $\mu_{q_1,\cdots,q_j;q}$. Define

$$\Pi_{t_N}^{(q_1,\cdots,q_j),N}, \tag{3.724}$$

as the total number of dual particles at time t_N starting the particle component η_t of the dual process $(\eta_t, \mathcal{F}_t^{++})$ with

$$q_i \text{ particles at distinct sites } i = 1,\cdots,j. \tag{3.725}$$

Then write

$$\mu_{q_1,\ldots,q_j;q}(t_N) = E\left[\exp\left(-\frac{m}{N}\int_0^{t_N} \Pi_s^{N,(q_1,\cdots,q_j)} ds\right)\right]. \tag{3.726}$$

q.e.d.

Part 2 *(Growth of dual clouds)*

Looking at the r.h.s. of formula (3.716)-(3.718) we see that for calculating higher moments we have to study the asymptotic (as $N \to \infty$) behaviour of the dual system starting with *more* than one particle. Of particular importance is to estimate the *overlap* in the dual particle population of different families due to different ancestors at the starting time of the evolution since this will allow to control the terms of order N^{-k} which are needed to compensate for the $\binom{N}{k}$ term on the r.h.s. of (3.716). This is addressed in the form needed for the calculation of first and second moments in the following definition and proposition.

Definition 3.23 *(Dual clouds for moment calculations)*

(a) To calculate first and second moments at one time we calculate the dual particle number asymptotically as follows if we start with one or two particles. We choose W_a as:

- $W_a = W_1$ *where* $W_1 = W^{1,1} = W$ *defined by (3.143) starting with one* $1_{\{1\}}$ *factor at time* 0
- $W_a = W^{1,2} = W_1 + W_2$ *where* W_1, W_2 *are independent copies of the random variable* W *arising from one factor at each of two different sites,*
- $W_a = W^{2,1}$ *which is the corresponding random variable when the CMJ process is started with the two factors* $1_{\{1\}} \otimes 1_{\{1\}}$ *at time* 0 *at the same site.*
- *Similarly we write* $W_a(\cdot)$ *for the function* $(W_a(t))_{t \geq 0}$ *arising from* $e^{-\alpha t} K_t^a$.

Let ξ be a time-inhomogeneous Poisson process on $[0, \infty)$ with intensity measure given by

$$\frac{(\alpha + \gamma)}{c} \frac{(W_a(s))^2 e^{2\alpha s}}{N} ds. \tag{3.727}$$

Note ξ depends on the chosen a but only through W_a.

Let

$$\widetilde{g}(u, s) \tag{3.728}$$

(which evolves independently of the parameter N) be the size of that green cloud of sites of age u produced by one collision (creating a red particle) at time s. Then $\widetilde{g}(t, s)$ has a law depending only on $t - s$, namely

$$\widetilde{g}(t - s) = e^{\alpha(t-s)} W^g(t - s), \tag{3.729}$$

where $W^g(t - s) \to W^g$ as $t - s \to \infty$ and W^g is defined by this relation.

Furthermore introduce for a realization of ξ a process $\mathfrak{g}$ depending on the parameters N and whose law depends on the choice of a but only through $W_a(\cdot)$ and which describes the size of the complete green population at time t:

$$\mathfrak{g}(W_a(\cdot), N, t) = \int_0^t \widetilde{g}(t, s) \xi(ds). \tag{3.730}$$

Analogously to $\widetilde{g}, \mathfrak{g}$ we define the pair

$$\widetilde{g}_{\text{tot}}, \mathfrak{g}_{\text{tot}} \tag{3.731}$$

as the number of particles in the green cloud, instead of the number of sites.

(b) To compute the joint second moment at two different times we consider

$$W_a = W_{j_1} + W_{j_2, s}, \tag{3.732}$$

which is the corresponding random variable when the CMJ process is started with the factor $1_{\{1\}}$ located at j_1 at time 0 and an additional factor $1_{\{1\}}$ is added at location j_2 at time $s > 0$.

Let $\xi^{(1,2)}$ be an inhomogeneous Poisson process on $[0,\infty)$ with intensity measure

$$(3.733) \qquad \left[(\alpha+\gamma)\frac{W_1(u)W_2(u)e^{2\alpha u}}{N}\right] du.$$

Then with

$$(3.734) \qquad t = t_N,$$

we consider time $t+s$. Furthermore set (recall (3.734))

$$(3.735) \qquad \mathfrak{g}_{(1,2)}(W_1(\cdot), W_2(\cdot), N, t, t+s) = \int_s^t \widetilde{g}(t_N, u)\xi^{(1,2)}(du).$$

We denote by $K_t^{N,(j_1,0),(j_2,s)}$ the number of occupied sites in the dual particle system starting with a particle at j_1 at time 0 and one at time j_2 at time s. □

Now we can state the main result on the growth of the dual clouds.

Proposition 3.24 *(Dual clouds)*
Let

$$(3.736) \qquad t = t_N, \text{ where } t_N \to \infty, \quad t_N = o(\frac{\log N}{2\alpha}),$$

more precisely,

$$(3.737) \qquad 2\alpha t_N - \log N \to -\infty \text{ as } N \to \infty.$$

Then consider the growth of the dual cloud in various initial states indicated by the subscript a.
(a) We have

$$(3.738) \qquad K_t^{N,(a)} = W_a(t)e^{\alpha t} - \mathfrak{g}(W_a(\cdot), N, t) + \mathcal{E}_a(N,t),$$

$$(3.739) \qquad \Pi_t^{N,(a)} = \frac{1}{c}(W_a)(t))(\alpha+\gamma)e^{\alpha t} - \mathfrak{g}_{\text{tot}}(W_a(\cdot), N, t) + \mathcal{E}_{a,\text{tot}}(N,t),$$

where we have for some $\kappa_2 \in (0,\infty)$

$$(3.740) \qquad \lim_{N\to\infty} Ne^{-2\alpha t_N} E[\mathfrak{g}_{\text{tot}}(W_a(\cdot), N, t_N)] = \kappa_2 E[(W_a)^2]$$

and the error term $\mathcal{E}_a$ or $\mathcal{E}_{a,\text{tot}}$ satisfies that

$$(3.741) \qquad \lim_{N\to\infty} E[Ne^{-2\alpha t_N}|\mathcal{E}_a(N,t_N)|] = 0.$$

(b) We have (as $N \to \infty$, recall (3.734)):

$$(3.742) \qquad K_{t+s}^{N,(j_1,0),(j_2,s)} \sim (W_1 e^{\alpha(t+s)} + W_2 e^{\alpha t}) - \mathfrak{g}(W_1(\cdot), W_2(\cdot), N, t, t+s) + O(\frac{e^{3\alpha t}}{N^2}),$$

and

$$(3.743) \qquad \Pi_{t+s}^{N,((j_1,0),(j_2,s))} = \frac{\alpha+\gamma}{c}(W_1 e^{\alpha(t+s)} + W_2 e^{\alpha(t+s)}) - \mathfrak{g}_{\text{tot}}(W_1(\cdot), W_2(\cdot), N, t, t+s) + O(\frac{e^{3\alpha t}}{N^2}),$$

where $\mathfrak{g}_{\text{tot}}(W_1(\cdot), W_2(\cdot), N, t, t+s)$ *satisfies the two conditions*

$$\begin{aligned}\mathfrak{g}_{\text{tot}}(W^{1,2}(\cdot), N, t, t+s) = \; & \mathfrak{g}_{\text{tot}}(W_1(\cdot), N, t+s) + \mathfrak{g}_{\text{tot}}(W_2(\cdot), N, t) \\ & + \mathfrak{g}_{(1,2),\text{tot}}(W_1(\cdot), W_2(\cdot), N, s, t),\end{aligned} \tag{3.744}$$

and

$$\lim_{N\to\infty} N \cdot e^{-\alpha(2t_N+s)} E[\mathfrak{g}_{(1,2),\text{tot}}(W_1(\cdot), W_2(\cdot), N, t_N, t_N+s)] = 2\kappa_2 E[W_1]E[W_2]. \tag{3.745}$$

Similarly such relations hold for $\mathfrak{g}$ *instead of* $\mathfrak{g}_{\text{tot}}$ *with* κ_2 *replaced by the appropriate different positive constant.*

(c) Now consider the case in which $N \to \infty$ *but* t *is independent of* N *(that is, we no longer take* $t_N \to \infty$*). Then we do not replace* $W_i(\cdot)$ *by* W_i *and* $\alpha(t), \gamma(t)$ *by* α, γ*. Then in (3.742), (3.743)* $W_1 \cdot (\alpha + \gamma)$ *is replaced by* $W_1(t+s)(\alpha(t+s) + \gamma(t+s))$ *and* $W_2 \cdot (\alpha + \gamma)$ *is replaced by* $W_2(t)(\alpha(t) + \gamma(t))$*.* □

Remark 44 *Note that in the expansion* $(W_1 + W_2)^2 = W_1^2 + 2W_1W_2 + W_2^2$ *we can associate the term* $2W_1W_2$ *with collisions of a particle from one cloud with sites occupied by the other cloud. In order to asymptotically evaluate these collision terms we could keep track of the white, red, green, blue particles from the two sites by labelling them, for example, by*

$$(W_-, R_-, G_-, B_-),\ (W_+, R_+, G_+, B_+), \tag{3.746}$$

respectively, and then considering the collisions of W_+ *with* W_-*, etc.*

Remark 45 *Note that the existence of* κ_2 *follows exactly in the same way as the argument leading up to (3.507) (proof of existence of* κ*) in Subsubsection 3.2.9. The only difference here is that it involves the growth of the number of green sites arising from either self collisions or between collisions of two different clouds.*

Proof of Proposition 3.24 (a) In order to justify (3.739) we now reconsider the development of the dual particle system on $\{1, \dots, N\}$ starting from one factor $1_{\{1\}}$ at location 1. We have used above the fact that for the range $0 \le t \le t_N$ where $\lim_{N\to\infty} \alpha \frac{t_N}{\log N} < 1$, the particle system is described asymptotically as $N \to \infty$ in first order by a CMJ process with Malthusian parameter α. Correction terms are analysed using the multicolour particles system.

Moreover if we follow two families starting with one factor $1_{\{1\}}$ at locations 1, 2 respectively, then the probability that they collide, that is occupy the same site in $\{1, \dots, N\}$, in this time regime goes to 0 as $N \to \infty$. This is essentially equivalent to the statement that the probability that a green particle is produced in the WRGB system in this time regime goes to zero as $N \to \infty$. However for finite N these probabilities are not zero and we now determine them by expansion in terms of powers of N^{-k}.

Recall that the system of white and black particles is a CMJ process as described above in which the number of occupied sites has the form

$$W(t)e^{\alpha t} \tag{3.747}$$

where $\lim_{t\to\infty} W(t) = W$ and this black and white system serves as an *upper bound* for the system of occupied sites in the dual process. Hence if at time t a migrant is produced by this collection following the dual dynamics, it moves to a randomly chosen point in $\{1, \dots, N\}$. Then the probability that it hits an occupied site is therefore at most:

$$\frac{W(t)e^{\alpha t}}{N}. \tag{3.748}$$

Therefore a upper bound for the process of collisions is given by an inhomogeneous Poisson process $\xi(s)$ with rate

$$(3.749)\qquad \frac{W^2(t)e^{2\alpha t}}{N}$$

and number of sites that have been hit this way in the time interval $[0,t]$ is at most

$$(3.750)\qquad \int_0^t \frac{W^2(s)e^{2\alpha s}}{N} ds \le \frac{1}{N}(\sup_t W(t))^2 \cdot \frac{1}{2\alpha}e^{2\alpha t} = \frac{1}{N}(W^*)^2 \frac{1}{2\alpha}e^{2\alpha t}.$$

Recall that $W^* = \sup\limits_{t \ge 0} W(t) < \infty$ a.s. For bounding second moments we can also show that $\sup_{t\ge 0} E[(W(t))^2] < \infty$. In particular the number of sites at which collisions take place by time t_N is $o(N)$, i.e. has spatial intensity zero as $N \to \infty$.

Moreover the expected number of sites to be hit more than once in the time interval $[0,t]$ goes to 0 as $N \to \infty$ at the order N^{-2}. This order holds also for red-red collision by migration in the multicolour particle representation.

Next we recall that when a white particle hits an occupied site it produces a red particle at this occupied site. Subsequently if the red particle is removed by coalescence with a white particle, then a green particle is produced. Due to the coalescence process eventually no red particles will remain at this site. Also due to migration eventually this site will revert to a pure white site. During the "lifetime" of such a multicolour site there is a positive probability that one or more green particles will migrate and then produce a growing cloud. (Recall that the lifetime has finite exponential moment.)

Once such a new site containing white and red particles is formed, a green particle is created by coalescence unless either the white or the red disappears before coalescence. The number of green particles to migrate from such a special site is random with probability law that depends on the number $\zeta(s)$ of the occupying white particles at the time s of arrival of the red particle (asymptotically as $N \to \infty$ no further red particle will arrive at this site before the coalescence or migration step of the red particle or disappearance of all whites at the site).

In other words there is a positive probability that the green particle will produce green migrants before it coalesces with a red particle (such that only one red particle remains and the green particle disappears (recall Remark 30)). We denote the probability $p^N(s)$ that at least one green migrant is produced during the lifetime of a special site due to a collision at time s satisfies

$$(3.751)\qquad p^N(s) \longrightarrow p(s) \text{ as } N \to \infty, \quad p(s) \longrightarrow p > 0 \text{ as } s \to \infty.$$

After the migration step the resulting green family, if non-empty, grows according to the CMJ process. Combining the two facts we see that the number of green sites this family occupies at time t, denoted $\widetilde{g}(t,s)$, satisfies:

$$(3.752)\qquad \widetilde{g}(t,s) = W^g(t,s)e^{\alpha(t-s)},$$

and $\lim_{t\to\infty} W^g(t,s) = W^g_s$ exists and converges to W^g as $s \to \infty$. Note that the event $\{W^g = 0\}$ occurs with the probability that a collision does not create a green migrant (see Remark 46 below) and otherwise

$$(3.753)\qquad W^g(t,s) := \text{ is the random growth constant for all the sites occupied by green migrants from a collision at a site at time } s \text{ in the interval } [s,t].$$

Also $W^g(t,r)$ is independent of $W(\cdot)$ and $W^g(t,r), W^g(t,s)$ are independent for $r \neq s$. The analogous quantity $W^g_{\text{tot}}(t,r)$ is defined to be the number of green particles produced in $[r,t]$ by a collision at time r.

Define the analog of $W^g(t,s)$ in the case where the white sites are in the stable age distribution as

$$W^{g,*}(t,s). \tag{3.754}$$

If we assume that the white population has reached the stable size distribution (for the number of white particles at an occupied site) at time s, then the probability law of the corresponding process $W^{g,*}(t,s)$ depends only on $t-s$.

Remark 46 *We note that at times $t \geq s_N$ with $s_N \to \infty$ the distribution of ζ in the CMJ process population approaches the stable age distribution as $N \to \infty$ uniformly in $t \geq s_N$.*

Then combining (3.750) and (3.752) we get that

$$E[\mathfrak{g}(W(\cdot),N,t)] \leq \int_0^t E[(W(s))^2]\frac{e^{2\alpha s}}{N}E[W^g(t,s)]e^{\alpha(t-s)}ds \leq \text{const}\cdot\frac{e^{2\alpha t}-e^{\alpha t}}{N}. \tag{3.755}$$

Provided the first inequality could be *modified* such that it becomes up to terms of order $o(N^{-1})$ an *equality*, the following limit exists

$$\lim_{N\to\infty} Ne^{-2\alpha t_N}E[\mathfrak{g}(W(\cdot),N,t_N)] = \kappa_2 E[W^2], \quad \kappa_2 = \int_0^\infty e^{-\alpha u}E[W^{g,*}(u)]du. \tag{3.756}$$

We note that the probability of collision in $[0,t_N]$ tends to 0 as $N\to\infty$ therefore we determine the probability of such a collision, the distribution of the collision time given that a collision occurs and the resulting number of green sites produced by this one collision. Given that a collision occurs the site at which it occurs is obtained by choosing a randomly occupied site.

In order to verify that such a modification exists and that the assumption on the error in the first inequality in (3.755) we have to (1) estimate how much we overestimate the collision rate by using the CMJ process (of black and white particles) instead of the real dual (i.e. (W,R,P) particles) and (2) in the application of W^g we need to control the error introduced by assuming that collision events involve sites with the stable age distribution for the white particles and replacing $\int_0^{t_N} e^{-\alpha(t_N-s)}E[W^g((t_N,s)]ds$ as $N\to\infty$ by the second part of the equation (3.756).

(1) We again note that the over estimate of the number of green particles corresponds to events having two or more collisions and this occurs with rate $\frac{e^{3\alpha t}}{N^2}$ and this correction as well as the exclusion of the purple particles produces a higher order error term.

(2) Consider the early collisions before reaching the stable age distribution. Noting that

$$\int_0^{s_N} e^{2\alpha s}e^{\alpha(t_N-s)}ds = Const\cdot e^{\alpha(t_N+s_N)} \tag{3.757}$$

therefore these contribute only a term increasing as $e^{\alpha t_N+s_N}$ whereas the latter ones grow at a rate $e^{2\alpha t_N}$. If $s_N\to\infty$, $s_N=o(t_N)$, then the early collisions form a negligible contribution. But then as $N\to\infty$, the white population approaches the stable age distribution by time s_N.

Note that for the late ones we cannot assume that $W^g(t,s)$ takes on its limiting value and for this reason we obtain the integral in the definition of κ_2. This proves Part (a) of Proposition 3.24.

(b) We now consider (3.743) when we begin with one $1_{\{1\}}$ factor at site j_1 at time 0 and one factor $1_{\{1\}}$ at $j_2\neq j_1$ at time $s>0$ and the total number of occupied sites at time t is what we called $\Pi_t^{N,(1,2)}$. This can be handled exactly as in (a). However we want to keep track of the subset of sites occupied by migrants coming from collisions between the two different families. For this reason we constructed the multicolour system.

Finally we have to show that the bound (3.788) gives actually asymptotically the correct answer. Following the argument in part (a) we note that the upper bound on the $G_{1,2}$ particles gives us

a lower bound on the white particles and this in turn gives us by (3.786) with a bootstrapping a lower bound on the number of $G_{1,2}$ particles. It then follows that the overestimate of the number of $G_{1,2}$ particles is no larger than const $\cdot \frac{\mathrm{e}^{3\alpha t_N}}{\mathrm{N}^2}$.

The analogous arguments also yields the claim (3.745) for $\mathfrak{g}_{(1,2),\mathrm{tot}}$.

(c) Other than including the time dependence for $W_i(\cdot)$ the proof follows the same lines as above. In particular the first two terms in (3.742) now correspond to the CMJ process obtained when $N = \infty$ and the limiting expected value of the correction term now has the form

$$\lim_{N\to\infty} N \cdot e^{-\alpha(2t+s)} E[\mathfrak{g}_{(1,2),\mathrm{tot}}(W_1(\cdot), W_2(\cdot), N, t, t+s)] = 2\kappa_2(t) E[W_1(t)] E[W_2(t)]. \tag{3.758}$$

This completes the proof of Proposition 3.24. ■

Part 3 *(Two dual clouds interacting)*

We want to compare the dual populations (note the different brackets and the $\sim$ indicating the populations instead of just the population sizes)

$$\widetilde{\Pi}^{N,\{j_1\}}, \widetilde{\Pi}^{N,\{j_2\}}, \widetilde{\Pi}^{N,\{j_1,j_2\}} \tag{3.759}$$

corresponding to the descendants of the particles at j_1, j_2, respectively evolving independently and the resulting population starting with both initial particles and evolving jointly. In order to carry out this comparison we construct a *coupling*, i.e. construct all three populations on one probability space in such a way that the respective laws are preserved. At the same time we want to be able to compare these populations with the respective collision-free versions evolving as CMJ-processes.

Construction of enriched multicolour system.

To construct this coupling we use again the WRGB-system and we construct the particle system starting with 1 particle at both sites j_1 and j_2, recall here the constructions from subsubsection 1.19. We now modify this coloured WRGB-particle system by marking the offspring of the initial particles at j_1, j_2 by using an enriched colour system, i.e. using the colours

$$\{W_1, R_1, G_1, B_1, P_1, W_2, R_2, G_2, B_2, P_2\}. \tag{3.760}$$

together with

$$\{W_1^*, W_2^*, R_1^*, R_2^*, G_1^*, G_2^*, B_1^*, B_2^*, P_1^*, P_2^*, R_{1,2}, G_{1,2}, B_{1,2}, P_{1,2}\}. \tag{3.761}$$

The second set of colours is reserved for particles involved in a collision between the two different families or their descendants.

The new features are

- When a W_1 particle hits a W_2 particle the result is a W_1^* and $R_{1,2}$ pair as well as the original W_2 particle at this site. A W_1^* particle does not coalesce with the W_2 particle.
- If the $R_{1,2}$ particle coalesces with the W_2 particle, the result is the pair consisting of a W_2 particle and a $G_{1,2}$ particle.
- The above holds with 1 and 2 interchanged.
- If W_1^* hits a W_1 particle it changes to a R_1^* particle.

With this expanded colour set, we now have to define a suitable multicolour dynamic. We define this system more precisely after outlining the goals of this construction. This is done so that the $(W_i, W_i^*, R_i, G_i),\ i = 1, 2$ describe the families in the absence of interactions between them. The particles $(R_{1,2}, G_{1,2})$ correspond to the particle families resulting from the interaction between the two families and $(W_1, W_2, R_1, R_2, G_1, G_2, R_{1,2}, G_{1,2})$ describe the interacting system. As before we want to couple the interacting system with a CMJ process.

Recall that the covariance of the type-1 mass in two locations if we write it in the form $E[XY]-E[X]E[Y]$ consists in the dual representation of a cloud of dual particles evolving jointly from two initial particles at different locations and two independently evolving clouds starting from one particle each. Hence the idea is that the covariance of $x_1^N(i,t+s)$ and $x_1^N(i,t)$ is determined by the particles in

$$\{\widetilde{\Pi}^{N,\{1\}} \cup \widetilde{\Pi}^{N,\{2\}}\} \backslash \widetilde{\Pi}^{N,\{1,2\}}, \tag{3.762}$$

where $\widetilde{\Pi}^{N,\{1\}}, \widetilde{\Pi}^{N,\{2\}}$ denotes the two independently evolving dual particle systems and $\widetilde{\Pi}^{\{1,2\}}$ the jointly evolving ones. The set $\widetilde{\Pi}^{N,\{1\}} \cup \widetilde{\Pi}^{N,\{2\}} \backslash \widetilde{\Pi}^{N,\{1,2\}}$ corresponds to the $G_{1,2}$ green particles.

We list now the precise evolution rules of the new coloured particle system. In the evolution of the populations arising from the two initial particles as before particles *migrate, coalesce* and *give birth* to new particles at the rates as in the dual. One modification is required, namely, in this case we allow singletons of one family to migrate to sites occupied by the other family and also singletons of one family in the presence of one or more particles of the other type can migrate. Here are the precise rules.

(0) The rules for the particles and colours appearing in (3.760) are exactly as before as long as the two populations (i.e. particles with a colour with one index either 1 or 2) do not interact or occupy the same site. If particles of the two families occupy the same site singletons of one family can now migrate.

Furthermore we have the four additional rules concerning the colours newly introduced in (3.761):

(1) When a white particle from family 1 and a white particle from family 2 coalesce the result is one W_1^*, one W_2^* particle and a $R_{1,2}$ particle. The W_ℓ^* particles evolve according to the rules of the dual particle system, but ignoring further collisions with the respective other population and upon collision with the own population a R_ℓ^*-particle is created replacing the W_ℓ^*-particle. Similarly proceed with the other $*$-colours.

(2) $R_{1,2}$ particles that coalesce with W_1 (respectively W_2) produces a W_1 (respectively W_2) and $G_{1,2}$ pair.
Two $R_{1,2}$ particles that hit an occupied site produce blue-purple pairs $(B_{1,2}, P_{1,2})$ where as before the $B_{1,2}$ particle is placed on the first empty site in the first copy of $\mathbb{N}$ continuing according to the collision-free dynamic and the $P_{1,2}$ particle behaves as a typical dual particle for $\widetilde{\Pi}^{N,\{1,2\}}$.

(3) Coalescence of $G_{1,2}$ and $R_{1,2}$ particles produce $R_{1,2}$ particles (cf. Remark 30).

(4) If a $R_{1,2}$ or $G_{1,2}$ particle more generally every colour in (3.761) gives birth the new particle carries the same type.

These set of rules uniquely define a pure jump Markov process on the state space describing the individuals with their colour and location and with the coupling information (recall (3.394) for how to formalize this).

The key properties of the new coloured dynamics are the following. On the same probability space we have the various systems we consider. Namely

$$\widetilde{\Pi}^{N,\{i\}} = \text{ union of the } W_i, W_i^*, R_i, P_i, R_i^*, P_i^*, \text{ for } i=1,2 \tag{3.763}$$

The new system also includes our dual starting with two particles, namely, (by construction)

$$\widetilde{\Pi}^{N,\{1,2\}} = \text{ the union of the } W_1, W_2, R_1, R_2, P_1, P_2, R_{1,2}, G_{1,2} \text{ particles.} \tag{3.764}$$

Furthermore we can again identify a CMJ-process,

$$\text{union of } (W_1, W_2, R_1, R_2, G_1, G_2, G_{1,2}, B_1, B_2, R_{1,2}, B_{1,2})\text{-particles} \widehat{=} \text{ CMJ process.} \tag{3.765}$$

Remark 47 *To compute higher moments a* complete *bookkeeping of the difference would require more and more different colours whenever such a collision with subsequent coalescence occurs. However although these collisions do occur, we note that they do not contribute to the calculation up to terms of order N^{-2}. Therefore we again ignore these higher order terms instead of introducing further colours.*

In the sequel we shall retain the W_1^* and W_2^* particles but not the $R_\ell^*, B_\ell^*, P_\ell^*$ particles since the order in N of these sets of particles are of too small to play a role for our purposes.

Lemma 3.25 *(Negligible colours)*
Let $\mathcal{N}_t^\ell$ denote the number of R_ℓ^*, B_ℓ^* or P_ℓ^* particles in the system ℓ. Then

$$E[\mathcal{N}_t^\ell] = O(\frac{e^{3\alpha t}}{N^2}). \qquad \square \tag{3.766}$$

Proof of Lemma 3.25 This follows since these correspond to families resulting from two or more collisions and these have order $O(\frac{e^{3\alpha t}}{N^2})$. q.e.d.

If we count only the occupation numbers by the various colours the state space is given by

$$\begin{aligned}&\mathcal{M}_c[(\{W_1, W_1^*, R_1, G_1, B_1\} \cup \\ &\qquad \{W_2, W_2^*, R_2, G_2, B_2\} \cup \{R_{1,2}, G_{1,2}, B_{1,2}, P_{1,2}\}) \times (\{\{1, 2, \dots, N\} + \mathbb{N} + \mathbb{N}\})]\end{aligned} \tag{3.767}$$

(where $\mathcal{M}_c$ denotes the set of counting measures).

This concludes the construction of a coupling via a new enriched enriched multicolour particle system.

Consequences of the coupling

Using the arguments of part (2) above we can conclude that:

$$\begin{aligned}\Pi_t^{N,\{1\}} + \Pi_t^{N,\{2\}} \sim \tfrac{1}{c}((W' + W''))(\alpha+\gamma)e^{\alpha t} - [\mathfrak{g}_{\text{tot}}((W'(\cdot)), N, t) + \mathfrak{g}_{\text{tot}}((W''(\cdot)), N, t)] \\ + \mathcal{E}_{1,\text{tot}}(N, t) + \mathcal{E}_{2,\text{tot}}(N, t),\end{aligned} \tag{3.768}$$

$$\Pi_t^{N,\{1,2\}} \sim \frac{1}{c}((W' + W''))(\alpha+\gamma)e^{\alpha t} - \mathfrak{g}_{\text{tot}}((W'(\cdot) + W''(\cdot)), N, t) + \mathcal{E}_{1,2;\text{tot}}(N, t), \tag{3.769}$$

where

$$\mathcal{E}_{\ell,\text{tot}}(N, t) = o(N^{-1}e^{2\alpha t_N}) \text{ as } N \to \infty, \quad \ell = 1, 2 \text{ or } (1, 2), \tag{3.770}$$

$$\mathfrak{g}_{\text{tot}}((W'(\cdot) + W''(\cdot)), N, t) = \int_0^{t_N} \widetilde{g}_{\text{tot}}(t_N, s)\xi^{(1,2)}(ds), \tag{3.771}$$

with $\xi^{(1,2)}$ a time inhomogeneous Poisson process on $[0, \infty)$ with intensity measure

$$\frac{((W'(\cdot)(s) + W''(\cdot)(s)))^2 e^{2\alpha s}}{N} ds \tag{3.772}$$

and $\widetilde{g}_{\text{tot}}(t, s)$ is the size of the green cloud at time t produced at time s by a *newly* created red particle at a white site.

We know furthermore that

$$E[\mathfrak{g}_{\mathrm{tot}}((W'+W'')(\cdot),N,t)] = \kappa_2 E[(W'+W'')^2]\frac{(e^{2\alpha t}-e^{\alpha t})}{N}. \tag{3.773}$$

Then combining all these properties (3.735) follows by comparing the expression for $\mathfrak{g}_{\mathrm{tot}}((W'+W'')(\cdot),N,t)$ with $\mathfrak{g}_{\mathrm{tot}}(W'(\cdot),N,t)+\mathfrak{g}_{\mathrm{tot}}(W''(\cdot),N,t)$.

Therefore the process corresponding to the quantity

$$\Pi_t^{N,1}+\Pi_t^{N,2}-\Pi_t^{N,\{1,2\}}, \quad t\geq 0 \tag{3.774}$$

has an upper bound which is generated by the Poisson process on $[0,\infty)$ with intensity measure

$$\left(\frac{(2W'W'')e^{2\alpha s}}{N}+O(e^{3\alpha s}N^{-2})\right)ds. \tag{3.775}$$

This corresponds to the difference term (i.e. lost particles due to collision and coalescence between the two families) given by the $G_{1,2}$-particles. It is bounded above by allowing the rate of production of collisions leading to the production of $G_{1,2}$ particles by the CMJ-process which excludes collisions. We can then obtain a lower bound by reducing the white families by the respective upper bounds for the sets of green particles as well as the blue particles. This replaces the white family $W(t)e^{\alpha t}$ by $W(t)e^{\alpha t}-O(\frac{e^{2\alpha t}}{N})$. Using this we can show that the result is an error term which is of order $O(\frac{e^{3\alpha t}}{N^2})$.

We now want a more detailed description in which we keep track explicitly of the collisions of the two families and the resulting $G_{1,2}$ families. We therefore consider

$$K_t^{N,\{1,2\}}, \Psi_t^N(i_{W_1},i_{W_2},i_{R_1},i_{R_2},i_{R_{1,2}},i_{G_1},i_{G_2},i_{G_{1,2}},i_{B_1},i_{B_2},i_{B_{1,2}},i_{B_{1,2}}), \tag{3.776}$$

which denotes the number of occupied sites at time t, respectively the number of sites having i_{W_1} white particles from the first population, i_{W_2} white particles from the second population, etc. Similarly we can define systems for subsets of colours as we did before. We also consider the pair (u^N,U^N) describing the new coloured particle system and a new corresponding limiting system (u,U) for $N\to\infty$. For the $(W_1,W_2,R_1,R_2,G_1,G_2,R_{1,2},G_{1,2},B_1,B_2)$-system we modify the equation for U_{WRGB}, namely, (3.459) to include the contribution to the dynamics of collisions between W_1 and W_2 particles. Similar we can proceed with other colour combinations.

Then we obtain the following representation formula. Here we use the convention to write

$$U_t^N(i_{W_j},i_{W_j^*},i_{R_j},i_{P_j}), U_t^N(i_{W_1},i_{W_2},i_{R_{1,2}}),\cdots \tag{3.777}$$

for $U_t^N(\cdot)$ all other colours not appearing in the argument summed out. Then:

$$\begin{aligned}\Pi_t^{N,\{j\}} &= u^N(t)\sum_{i_{W_j},i_{W_j^*},i_{R_j},\cdots,i_{P_j^*}}(i_{W_j}+i_{W_j^*}+i_{R_j}+i_{P_j}+i_{R_j^*}+i_{P_j^*})U_t^N(i_{W_j},i_{W_j^*},i_{R_j},i_{P_j},i_{R_j^*},i_{P_j^*}),\\ &\qquad j=1,2,\end{aligned} \tag{3.778}$$

$$\begin{aligned}\Pi_t^{N,\{1,2\}} &= u^N(t)\sum_{i_{W_1},\cdots,i_{P_{1,2}}}(i_{W_1}+i_{W_2}+i_{R_1}+i_{R_2}+i_{R_{1,2}}+i_{P_1}+i_{P_2}+i_{P_{1,2}})\\ &\qquad U_t^N(i_{W_1},i_{W_2},i_{R_1},i_{R_2},i_{R_{1,2}},i_{P_1},i_{P_2},i_{P_{1,2}}),\end{aligned} \tag{3.779}$$

where the sum is over $i_{W_1},i_{W_2},i_{R_1},i_{R_2},i_{R_{1,2}},i_{P_1},i_{P_2},i_{P_{1,2}}$.

We now focus on the are sites at which a collision occurs between a W_1 particle and a W_2 site (or vice versa) which can then produce migrating $R_{1,2}$ or $G_{1,2}$-particles. Therefore we define as *"special" sites* at time t, those sites at which

(3.780) one or more $R_{1,2}$ particles are present at time t and *also* a W_1 or W_2-particle is present at time t.

Note that special sites have a finite lifetime and can produce migrating $R_{1,2}$, $G_{1,2}$ particles during their lifetime. Denote the number of special sites by:

$$K_t^{N,*} = \left(\sum_{i_{W_1}, i_{W_2}, i_{R_{1,2}}} (1_{i_{R_{1,2}} \geq 1} \cdot (1_{i_{W_1} \neq 0} \vee 1_{i_{W_2} \neq 0}) U_t^N(i_{W_1}, i_{W_2}, i_{R_{1,2}}) \right) \cdot u^N(t). \tag{3.781}$$

In the calculation of second moments we can ignore errors of size $o(N^{-2})$, we can work with the system of $W_1, W_2, R_1, R_2, R_{1,2}$ and ignore the purple particles i.e. the colours $P_1, P_2, P_{1,2}$. Therefore it suffices to determine the number of sites occupied by $W_1, W_2, R_1, R_2, R_{1,2}$ particles which estimates $K_{t+s}^{N,(j_1,0),(j_2,s)}$ up to errors of order $O(\frac{e^{3\alpha t}}{N^2})$. This results in the expansion given in (3.742) where in (3.744) the three $\mathfrak{g}_{\text{tot}}$ terms correspond to the sites occupied by only G_1, G_2 and $G_{1,2}$ particles. It remain to estimate these terms. The G_1, G_2 terms are similar to those considered earlier so we focus on the $G_{1,2}$ sites and particles. We will see below that key to the determining the asymptotics of the covariance at two sites as $N \to \infty$ is to analyse the number of $G_{1,2}$-particles asymptotically as $N \to \infty$. We have now all the tools to start the estimation.

Estimates on the number of $G_{1,2}$-particles

In order to study the asymptotics of the *number of $G_{1,2}$ particles* at time t_N, in sublogarithmic time scales which we denote this number by:

$$\Pi_{t_N}^{N,G_{1,2}} \text{ with } t_N \text{ as in (3.736).} \tag{3.782}$$

We next recall that *independent* systems of dual particle systems with starting particles in site j_1 at time 0 and site j_2, at time $s > 0$ with $j_1 \neq j_2$, produce clouds as $N \to \infty$ with independent stable size distributions and the number of sites of the two dual populations is denoted by

$$K_t^{N,\{1\}}, K_t^{N,\{2\}}. \tag{3.783}$$

We then have by the CMJ-theory and the K_t for K_t^N approximation $N \to \infty$,

$$K_{t_N}^{N,\{1\}} \sim W' e^{\alpha t_N}, \quad K_{t_N}^{N,\{2\}} \sim W'' e^{\alpha t_N}. \tag{3.784}$$

To identify the correction terms needed for second moment calculations we need to consider the dual particle system starting with two particles one at site 1, one at site 2 and estimate the effects of the *collisions between the two clouds*. Using our embedding in the coloured particle system this can involve a collision in which a W_1 particle can migrate to a site already occupied by a W_2 particle (or vice versa). Once a collision occurs, this produces a $W_1^*, R_{1,2}$ pair. The $R_{1,2}$ particle can move to another site before it coalesces with one of the W_2 particles of the other colour. If it coalesces the W_2 particle remains an a new $G_{1,2}$-particle is added at this site. The $G_{1,2}$ particles can then reproduce and coalesce with other particles. This way they can produce more $G_{1,2}$ particles or migrate to produce a new $G_{1,2}$ site. The system of $G_{1,2}$ particles after the creation of a $G_{1,2}$-particle has the property that after foundation a $G_{1,2}$-family does not anymore depend on the other colours, therefore the $G_{1,2}$-system evolves as a copy of the basic one type *CMJ particle system* with *immigration* given by a randomly fluctuating source which is independent of

the CMJ-process. The creation of new $R_{1,2}$ particles at time s is determined by the number of (W_1, W_2)-pairs at time s.

Suppose we are given the evolution of the W_1, W_2-particles. Then the events of collisions between W_1 and W_2 particles is given by a Poisson process and when a collision occurs at a site there is a positive probability of a coalescence and therefore the production of a green particle. This then produces a growing population of $G_{1,2}$ descendants according to the CMJ-theory leading to the analogue expression to (3.755) which we had for W_1, W_2-particles.

If we replace the system of white particles of the two types by the *independent* site-1 and site-2 system, we obtain an *upper* bound for the mean number of $G_{1,2}$ particles given the growth constants W_1 and W_2 of the two clouds, so that at time $t = t_N$ provided that

$$(3.785) \quad t_N = o(\frac{\log N}{\alpha}),$$

by just calculating the mean.

This results in the following asymptotic population size for the $G_{1,2}$ particles which is then again given by the behaviour of the conditional mean (given the W_1, W_2 populations):

$$(3.786) \quad \frac{c}{N} \int_0^{t_N} E[W] e^{\alpha(t_N - u)} (\Pi_u^{N,\{1\}} K_u^{N,\{2\}} + \Pi_u^{N,\{2\}} K_u^{N,\{1\}}) p^N(u) du + o(N^{-1}).$$

The real system requires a correction term due to the interaction of the two clouds leading to a reduction of the two independent white systems by the $G_{1,2}$ but which is of lower order in N than the W_1, W_2-particle numbers and their effect, i.e. $o(N)$. Hence from (3.786) we get the asymptotic upper bound for the mean number of $G_{1,2}$ particles as $N \to \infty$ if we condition on the growth constants W' and W'' of the two clouds. As bound we get an expression which is asymptotically $(N \to \infty)$ equal to:

$$(3.787) \quad \sim \frac{\kappa_2^N}{N} W' W'' e^{2\alpha t_N},$$

and which differs from the real system by an error term of order $o(\frac{e^{2\alpha t_N}}{N})$.

In fact we can obtain

$$(3.788) \quad \lim_{N \to \infty} N e^{-2\alpha t_N} E[\mathfrak{g}_{(1,2)}(W(\cdot), N, t_N)] = \kappa_2 E[W'] \cdot E[W''], \quad \kappa_2 = \int_0^\infty e^{-\alpha u} E[W^{g,*}(u)] du.$$

exactly the same way as for (3.756).

Remark 48 *In the longer time regime $\frac{\log N}{\alpha} + t$ the growing cloud of green particles $G_{1,2}$ satisfies a law of large number effect similar as in (3.485), that is, conditioned on W_1, W_2, the variance of the normalized number of $G_{1,2}$-particles goes to zero as $N \to \infty$.*

Part 4 *(Random exponentials)*

We next present the facts needed about the expansion of random exponentials in the following lemma.

Lemma 3.26 *(Asymptotics of random exponentials)*

(a) Let X be a nonnegative random variable satisfying $E[X^k] < \infty$ for all $k \in \mathbb{N}$. Let $f(\varepsilon, X)$ be such that

$$(3.789) \quad f(\varepsilon, X) \geq 0, \quad f(\varepsilon, X) \sim \varepsilon[X - \varepsilon\kappa X^2 + O(\varepsilon^2) X^3],$$

that is,

$$\sup_{0<\varepsilon\le 1} \frac{|f(\varepsilon,x)-\varepsilon X+\varepsilon^2X^2|}{\varepsilon^3}<\infty. \tag{3.790}$$

Consider

$$E[1-e^{-f(\varepsilon,X)}]. \tag{3.791}$$

Then the following two relations hold:

$$\lim_{\varepsilon\to 0}\frac{E[1-e^{-f(\varepsilon,X)}]}{\varepsilon}=E[X], \tag{3.792}$$

$$\lim_{\varepsilon\to 0}\frac{E[1-e^{-f(\varepsilon,X)}-\varepsilon X]}{\varepsilon^2}=(1+2\kappa)E[X^2]. \tag{3.793}$$

(b) Consider now random variables X,Y with $E[Y^k+X^k]<\infty$ for all $k\in\mathbb{N}$. Let Y_ε be a superposition of a Poisson number of independent copies of a random variable Y with intensity εX^2. Assume now that $f(\varepsilon,X,Y)\ge 0$ and that we have:

$$f(\varepsilon,X,Y)\sim\varepsilon[X-Y_\varepsilon]. \tag{3.794}$$

Then (3.792) and (3.793) are satisfied with

$$f(\varepsilon,X) \text{ replaced by } f(\varepsilon,X,Y) \text{ and } 1+2\kappa \text{ by } 1+EY. \qquad \square \tag{3.795}$$

Proof (a) Since the limits (3.792) and (3.793) are not affected by the $O(\varepsilon^2)$ term in (3.789) we can assume that it is a constant without loss of generality. Then by the mean value theorem,

$$\frac{1-e^{-\varepsilon X+\varepsilon^2X^2-\varepsilon^3X^3}}{\varepsilon}=D(X,\varepsilon^*) \text{ where } D(X,\varepsilon)=\frac{d}{d\varepsilon}[1-e^{-\varepsilon X+\varepsilon^2\kappa X^2-\varepsilon^3X^3}] \tag{3.796}$$

and where $0<\varepsilon^*\le\varepsilon$ and $\lim_{\varepsilon\to 0}D(X,\varepsilon)=X$. Therefore we have for $c>0$ that

$$E\left[\frac{1-e^{-\varepsilon X+\varepsilon^2\kappa X^2-\varepsilon^3X^3}}{\varepsilon}\right]=E[D(X,\varepsilon^*)1_{(X\le\frac{c}{\varepsilon})}]+\frac{\text{const}}{\varepsilon}\cdot P[X>\frac{c}{\varepsilon}]. \tag{3.797}$$

Then

$$D(X,\varepsilon)=\frac{d}{d\varepsilon}[1-e^{-\varepsilon X+\varepsilon^2\kappa X^2-\varepsilon^3X^3}]=[(X-2\varepsilon^2\kappa X^2+3\varepsilon^2X^3)e^{-\varepsilon X+\varepsilon^2\kappa X^2-\varepsilon^3X^3}]. \tag{3.798}$$

Note that

$$E[\sup_{0<\varepsilon\le 1,0\le X\le\frac{c}{\varepsilon}}|D(X,\varepsilon)|]\le \text{const}E[X+X^2+X^3]<\infty. \tag{3.799}$$

Therefore by dominated convergence for every $c>0$:

$$\lim_{\varepsilon\to 0}E[1_{(X\le\frac{c}{\varepsilon})}\cdot D(X,\varepsilon)]=E[X]. \tag{3.800}$$

Since $E[X^k]<\infty$ for $k\in\mathbb{N}$ we have

$$P[X>\frac{c}{\varepsilon}]\le E[X^k]\cdot(\frac{\varepsilon}{c})^k. \tag{3.801}$$

Combining (3.800), (3.801) we conclude from (3.797) the claim (3.792).

Similarly for the second order expansion write ,

$$\frac{1}{\varepsilon}[\frac{1-e^{-\varepsilon X+\varepsilon^2\kappa X^2-\varepsilon^3 X^3}}{\varepsilon}-X]=\frac{1}{\varepsilon}[D(X,\varepsilon^*)-D(X,0)]=D^2(X,\varepsilon^{**}), \tag{3.802}$$

where $D^2(X,\varepsilon)=\frac{d^2}{d\varepsilon^2}[1-e^{-\varepsilon X+\varepsilon^2\kappa X^2-\varepsilon^3X^3}]$ and

$$\lim_{\varepsilon\to 0}[D^2(X,\varepsilon)]=X^2+2\kappa X^2=(1+2\kappa)X^2. \tag{3.803}$$

As above we then conclude the claim (3.793).

(b) We have

$$1-e^{-f(\varepsilon,X,Y)}=1-e^{-\varepsilon X+\varepsilon Y}1_{Y\neq 0}-e^{-\varepsilon X}1_{Y=0} \tag{3.804}$$

and

$$\sup_{0<\varepsilon\leq 1,\ Y<\frac{1}{\varepsilon}}|\frac{d}{d\varepsilon}(1-e^{-f(\varepsilon,X,Y)})|\leq \text{const}\cdot(X+Y). \tag{3.805}$$

Therefore the relation is now obtained as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{E[1-e^{-f(\varepsilon,X,Y)}]-\varepsilon E[X]}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\left[\frac{Er[((1-e^{-\varepsilon X}])+\varepsilon E[(1-e^{-\varepsilon X^2})]E[Y1_{0<Y\leq\frac{1}{\varepsilon}}]-\varepsilon E[X]}{\varepsilon^2}\right]\\
&=\lim_{\varepsilon\to 0}\left[\frac{\varepsilon^2E[X^2]+\varepsilon^2E[X^2]E[Y]+P[Y>\frac{1}{\varepsilon}]}{\varepsilon^2}\right]=E[X^2](1+E[Y]).
\end{aligned} \tag{3.806}$$

■

Part 5: *(Proof of Proposition 3.21)*

To prove (a) (3.710) and (3.711) it suffices to show that

$$0<\lim_{N\to\infty}E[e^{-\alpha t_N}\widehat{x}_2^N(t_N)]<\infty,\quad 0<\lim_{N\to\infty}\text{Var}[e^{-\alpha t_N}\widehat{x}_2^N(t_N)]<\infty, \tag{3.807}$$

$$\limsup_{N\to\infty}E[(e^{-\alpha t_N}\widehat{x}_2^N(t_N))^3]<\infty, \tag{3.808}$$

so that the family $\{\mathcal{L}(\widehat{x}_2^N(t_N)):N\in\mathbb{N}\}$ is tight and their limit points are non-degenerate and random since they have finite non-zero mean and variance. If we can identify the limits in (3.807), then we can obtain (3.712) and (3.713) via (3.808).

The proof proceeds in steps, using duality representation of general moments in order to obtain first, second and then third moments of $\widehat{x}_2^N(t_N)$. It follows from Lemma 3.22 that we must consider *correction terms* of order $\frac{1}{N^k}$ in the calculation of kth moments of $\widehat{x}_1^N(t)$ in order to later derive the asymptotics of the moments of $\widehat{x}_2^N$ using the representation of formula (3.716).

Recall that the basic dual relation

$$E[(x_2^N(i,t))^a]=E\left[1-\exp\left(-\frac{m}{N}\int_0^{t_N}\Pi_s^{N,a}ds\right)\right] \tag{3.809}$$

where $\Pi_s^{N,a}$ denotes the total number of dual particles if we start with $a\in\mathbb{N}$ particles at site $i\in\{1,\ldots,N\}$. We have obtained a representation of this set of particles in terms of a decomposition into white, red and purple particles embedded in a multi-colour particle system of white, red,

purple, blue and green particles (WRGPB-system). We have also coupled this system to a related CMJ process which in particular provides an upper bound, which was the WRGB-system. We now look at this system in more detail in the time interval $[0, t_N]$ where t_N satisfies the hypothesis (3.709). We first recall that the number of occupied sites in the CMJ has the form

$$\Pi_s^{CMJ,a} = W_a(s)e^{\alpha s}, \quad \lim_{s\to\infty} W_a^s = W_a, \text{ a.s.} \tag{3.810}$$

Moreover the set of white and red particles in given by removing a set of green and blue particles from the CMJ process and we can bound the number of purple particles by the set of blue particles. Therefore we can represent the dual population by removing the green and blue particles with an error given by the difference between the blue and purple particles. Note that the blue and purple particles result from the event of *2 or more collisions* and therefore the error term can be obtained by determining this probability. Moreover we have seen that the collision process is given by a Poisson process with intensity bounded by $\alpha(s)W_a(s)e^{\alpha s}\frac{W_a(s)e^{\alpha s}}{N}$. Therefore the probability of a collision in $[0, t_N]$ is $O(\frac{e^{2\alpha t_N}}{N})$ and this goes to 0 as $N \to \infty$ as $N \to \infty$ under hypothesis (3.709). Let # denote the number of collisions in $[0, t_N]$. Moreover the probability of the event two or more collisions is

$$P(\# \geq 2) \leq Z_1\left(\frac{e^{4\alpha t_N}}{N^2}\right), \tag{3.811}$$

where Z_1 is a random variable having finite moments. Conditioned on the presence of a collision at time $s \in [0, t_N]$ the resulting set of green particles is given by $W^g_{\text{tot}}(t_N, s)e^{\alpha(t_N - s)}$ where $W^g_{\text{tot}}(t_N, s)$ is a nonnegative random variable with all moments and bounded by an independent copy of the CMJ process.

Then using Taylor's formula with remainder we obtain

$$\begin{aligned} &E\left[1 - \exp\left(-\frac{m}{N}\int_0^{t_N}\Pi^{N,a}(s)ds\right) - \left(\frac{m}{N}\left[\int_0^{t_N}\Pi^{N,a}(s)ds\right]\right)\right] \\ &= -\frac{m^2}{2N^2}E\left[\left(\int_0^{t_N}\Pi^{N,a}(s)ds\right)\right]^2 + E[\mathcal{E}_2(N)] \end{aligned} \tag{3.812}$$

$$\mathcal{E}_2(N) \leq \frac{1}{6}\left(-\frac{m}{N}\left[\int_0^{t_N}\Pi^{CMJ,a}(s)ds\right]\right)^3 \leq \frac{m^3}{6N^3}Z_2e^{3\alpha t_N}, \tag{3.813}$$

where Z_2 has finite moments. Calculate first the expectation of the occupation integral.

$$\begin{aligned} &E\left[\frac{m}{N}\left[\int_0^{t_N}\Pi^{N,a}(s)ds\right]\right] \\ &= \frac{m}{N}E\left[\int_0^{t_N}\frac{(\alpha(s)+\gamma(s))}{c}W_a(s)e^{\alpha s}ds - \frac{1}{N}\int_0^{t_N}\frac{(\alpha(s)+\gamma(s))}{c}(W_a(s))^2W_{g,\text{tot}}(t_N, s)ds\right] \\ &\quad + E[\mathcal{E}_1(N)] \\ &\mathcal{E}_1(N) \leq Z_1\frac{e^{4\alpha t_N}}{N^3}. \end{aligned} \tag{3.814}$$

Here the second term comes from integrating over the event $\{\# = 1\}$ and corresponds to a green family produced by a single collision. To estimate the first term on the r.h.s. of (3.812) we calculate:

$$E\left[\frac{m}{N}\left(\int_0^{t_N}\Pi^{N,a}(s)ds\right)\right]^2 \tag{3.815}$$
$$= \frac{m^2}{N^2}E\left[\int_0^{t_N}\frac{(\alpha(s)+\gamma(s))}{c}W_a(s)e^{\alpha s}ds\right]^2 + E(\mathcal{E}_3(N))$$
$$\mathcal{E}_3(N) \le Z_3\frac{e^{3\alpha t_N}}{N^3}.$$

We are now ready to obtain study the first and second moments of $\widehat{x}_2(t_N)$. Consider a sequence $s_N \to \infty$ $0 \le t_N - s_N \to \infty$. Then using the basic properties of the CMJ process (3.143) and (3.166) we have $W_a(s) \to W_a, \alpha(s) \to \alpha$ and $\gamma(s) \to \gamma$ (a.s. and in L^1) as $s \to \infty$. Using this we obtain

$$\lim_{N\to\infty} Ne^{-\alpha t_N}E\left[\int_0^{t_N}\Pi_s^{N,a}ds\right] \tag{3.816}$$
$$= \lim_{N\to\infty} Ne^{-\alpha t_N}E\left[\int_0^{t_N}\frac{\alpha(s)+\gamma(s)}{c}W_a(s)e^{\alpha s}ds\right]$$
$$= \lim_{N\to\infty} Ne^{-\alpha t_N}\left(E\left[\int_0^{s_N}\frac{\alpha(s)+\gamma(s)}{c}W_a(s)e^{\alpha s}ds\right] + E\left[\int_{s_N}^{t_N}\frac{\alpha(s)+\gamma(s)}{c}W_a(s)e^{\alpha s}ds\right]\right)$$
$$= \frac{(\alpha+\gamma)}{c}W_a.$$

Therefore

$$\lim_{N\to\infty} Ne^{-\alpha t_N}\left(1 - E[\exp(-\frac{m}{N}\int_0^{t_N}\Pi_s^{N,a}ds)]\right) \tag{3.817}$$
$$= \lim_{N\to\infty} Ne^{-\alpha t_N}E\left[m\int_0^{t_N}\Pi_s^{N,a}ds\right]$$
$$= m^*E[W_a] \tag{3.818}$$

where $m^* = \frac{\alpha+\gamma}{\alpha c}$.

In the same way we obtain

$$\lim_{N\to\infty} N^2e^{-2\alpha t_N}\left(1 - E\left[\exp\left(-\frac{m}{N}\int_0^{t_N}\Pi_s^{N,a}ds\right) - \frac{m}{N}\int_0^{t_N}\Pi_s^{N,a}ds\right]\right) \tag{3.819}$$
$$= \kappa_2E[(W_a)^2] + \frac{(m^*)^2}{2}(E[W_a])^2.$$

Step 1 First moment of $\widehat{x}_2(t_N)$

Note that by exchangeability

$$E[\widehat{x}_1^N(t)] = NE[x_1^N(t)]. \tag{3.820}$$

Then

$$E[\widehat{x}_2^N(t)] = N - NE[x_1^N(t)]. \tag{3.821}$$

Using the above calculations we have

$$E[\widehat{x}_2^N(t_N)] = NE[x_2^N(t_N)] = NE\left[1 - \exp\left(-\frac{m}{N}\int_0^{t_N} \Pi_s^{N,a} ds\right)\right]. \tag{3.822}$$

Therefore using (3.817), we obtain

$$\lim_{N\to\infty} E[e^{-\alpha t_N} x_2^N(t_N)] = m^* E[W_a]. \tag{3.823}$$

Step 2 Second moment of $\widehat{x}_2(t_N)$

We will focus next on the calculation of the second moment. We follow the method used above but leave out some details.

To compute the second moment of $\widehat{x}_2^N(t)$ note that

$$\begin{aligned} E[(\widehat{x}_2^N(t)]^2) &= N^2 - 2NE[\widehat{x}_1^N(t)] + E[(\widehat{x}_1^N(t))^2] \\ &= N^2 - 2N^2 E[x_1^N(i,t)] + E[(\widehat{x}_1^N(t))^2] \end{aligned} \tag{3.824}$$

and with $i \neq j$

$$E[(\widehat{x}_1^N(t))^2] = NE[(x_1^N(i,t))^2] + N(N-1)E[x_1^N(i,t)x_1^N(j,t)]. \tag{3.825}$$

Since the formula contains $O(N^2)$ factors, in order to compute the r.h.s. up to order 1, as $N \to \infty$, it is necessary to include corrections of order $\frac{1}{N^2}$ in both $E[x_1^N(i,t)]$ and $E[x_1^N(i,t)x_1^N(j,t)]$, etc. The first moment expansion we have done in Step 1. This means that we have to consider corrections due to possible *intersections* of different clouds (clouds meaning the descendants of one of the initial individuals.)

Consider two clouds starting with one factor at different sites in the dual population which occupy the following number of sites respectively

$$W_{1,N}e^{\alpha t_N} \text{ and } W_{2,N}e^{\alpha t_N}, \text{ as } N \to \infty, \tag{3.826}$$

with t_N as in (3.709) and where $W_{i,N} = W_i(t_N) \to W_i$ as $N \to \infty$ for $i = 1, 2$. Then the rate at which the first cloud produces particles that migrate onto one of the sites occupied by the second cloud and symmetrically is as $N \to \infty$:

$$\frac{2(\alpha+\gamma)^2}{c}\frac{W_{1,N}W_{2,N}e^{2\alpha t_N}}{N} + o\left(\frac{e^{2\alpha t_N}}{N}\right) = \frac{2(\alpha+\gamma)^2}{c}\frac{W_1 W_2 e^{2\alpha t_N}}{N} + o\left(\frac{e^{2\alpha t_N}}{N}\right). \tag{3.827}$$

If the two clouds have an intersection, coalescence can reduce the number of particles compared to $(W_1+W_2)(\alpha+\gamma)e^{\alpha t_N}$ and this produces a *correction term* corresponding to the $G_{1,2}$ particles. We have analysed this correction term in all detail (in the proof of Proposition 3.24) using a coloured particle system but here using that construction and its consequences, recall (3.739-3.741), we get a result on the behaviour of the number of dual particles and can perform moment calculations. We use these results to calculate covariances for $i \neq j$, namely, we use the dual and expand the time integral using (3.743) with $s = 0$ to represent $\Pi_u^{N,(1,2)}$ for $u \in [0, t_N]$. We estimate $E[\int_0^{t_N} \mathfrak{g}_{\text{tot}}(W^{1,2}(\cdot), N, u)du]$ by $\frac{\kappa_2}{2c\alpha}E[(W^{1,2})^2]\left(\frac{e^{2\alpha t_N} - 2e^{\alpha t_N}}{N^2}\right)$ and (3.741) to bound the error term.

If t_N satisfies the above growth assumption then as $N \to \infty$ we can expand the exponential and again verify that we can take expectation in this expansion, as in (3.823). (Recall Lemma 3.26 for the justification of these expansions.) Then using again (3.809) we calculate as follows.

$$\begin{aligned}
&E[x_1^N(i,t_N)x_1^N(j,t_N)] \\
&= E\left[e^{-\frac{m^*}{N}\left[(W_1+W_2)(\alpha+\gamma)(e^{\alpha t_N}-1)-\int_0^{t_N}\mathfrak{g}_{\mathrm{tot}}((W_1+W_2)(\cdot),N,u)-e^{\alpha t_N}du+o(N^{-1})\right]}\right] \\
&= 1-\tfrac{m^*}{N}E[(W_1+W_2)](\alpha+\gamma)(e^{\alpha t_N}-1)+\tfrac{m\kappa_2}{2c\alpha}E[(W^{1,2})^2]\tfrac{e^{2\alpha t_N}}{N^2} \\
&\qquad\qquad +\tfrac{(m^*)^2}{2N^2}(E[W_1+W_2])^2e^{2\alpha t_N}+o\left(\tfrac{e^{2\alpha t_N}}{N^2}\right).
\end{aligned} \tag{3.828}$$

We have as $N \to \infty$ for $i \neq j$:

$$\begin{aligned}
&E[x_1^N(i,t_N)x_1^N(j,t_N)] \\
&\sim 1-\frac{m^*}{N}E[W_1+W_2]e^{\alpha t_N} \\
&+\frac{m\kappa_2}{2c\alpha}E[(W^{1,2})^2]\frac{e^{2\alpha t_N}}{N^2}+\frac{(m^*)^2}{2N^2}(E[W_1+W_2])^2e^{2\alpha t_N}+o(\frac{e^{2\alpha t_N}}{N^2}),
\end{aligned} \tag{3.829}$$

where $W^{2,1}$ is the growth constant if we start with *two* particles at *one* site. Similarly we get

$$\begin{aligned}
&E[(x_1^N(i,t_N))^2] \\
&\sim 1-\frac{m^*}{N}E[W^{2,1}]e^{\alpha t_N} \\
&+\frac{m\kappa_2}{2c\alpha}E[(W^{2,1})^2]\frac{e^{2\alpha t_N}}{N^2}+\frac{(m^*)^2}{2N^2}(E[W^{2,1}])^2e^{2\alpha t_N}+o(\frac{e^{2\alpha t_N}}{N^2}).
\end{aligned} \tag{3.830}$$

Furthermore note that

$$\mathrm{Var}\left(\widehat{x}_2^N(i,t_N)\right)=\sum_{i=1}^N \mathrm{Var}(x_1^N(i,t_N))+\sum_{i\neq j}^N \mathrm{Cov}(x_1^N(i,t_N),x_1^N(j,t_N)). \tag{3.831}$$

Then by (3.825) combined with (3.809) and (3.828) the total mass of type 2 satisfies for $N \to \infty$ by expanding the exponential (below $i \neq j$):

$$\begin{aligned}
E[(\widehat{x}_2^N(t_N))^2] &= N^2-2NE[\widehat{x}_1^N(t_N)]+E[(\widehat{x}_1^N(t_N))^2] \\
&= N^2-2N^2E[x_1^N(i,t)]+NE[(x_1^N(i,t_N))^2]+N(N-1)E[x_1^N(i,t_N)x_1^N(j,t_N)] \\
&= -N^2+2Nm^*E[W_1]e^{\alpha t_N}+N(1-\tfrac{m^*}{N}E[W^{(2,1)}]e^{\alpha t_N})-m^2E[(W_1)^2]e^{2\alpha t_N} \\
&+N(N-1)(1-\tfrac{m^*}{N}2E[W_1]e^{\alpha t_N}+\tfrac{(E[(W_1+W_2)^2]\kappa_2 m^* e^{2\alpha t_N})}{N^2}) \\
&+\tfrac{1}{2}N(N-1)\tfrac{m^{*2}}{N^2}E[(W_1+W_2)^2]e^{2\alpha t_N}.
\end{aligned} \tag{3.832}$$

Therefore we get

$$\begin{aligned}
(\text{ r.h.s. } (3.832)\ ) \\
= \quad & 2E[W_1W_2]\kappa_2 m^* e^{2\alpha t_N}-m^*E[W^{(2,1)}]e^{\alpha t_N} \\
& +m^{*2}(E[W_1])^2e^{2\alpha t_N}+o(1),\ \text{as } N\to\infty.
\end{aligned} \tag{3.833}$$

This implies

$$\lim_{N\to\infty}(e^{-2\alpha t_N}E[(\widehat{x}_2^N(t_N))^2]) = (2\kappa_2 m^* + m^{*2})(E[W_1])^2, \tag{3.834}$$

$$\lim_{N\to\infty}(e^{-2\alpha t_N}Var[(\widehat{x}_2^N(t_N))]) = 2\kappa_2 m^*(E[W_1])^2. \tag{3.835}$$

We now consider the case with two times t_N and $t_N + t$. Following the same steps as above we can show that

$$\lim_{N\to\infty}(e^{-2\alpha t_N}E[(\widehat{x}_2^N(t_N))\widehat{x}_2^N(t_N+t)]) = (2\kappa_2 m^* + m^{*2})(E[W_1)]^2 e^{\alpha t}. \tag{3.836}$$

Step 3 (Third moments)

We can proceed again as before, expand the $x_2^N(t)$ in terms of $x_1^N(t)$ and apply the dual representation again. Here however we only need an upper bound and therefore we can bound the total number of dual particles from above by the ones of the collision-free process. Then the claim follows again easily by using (3.147).

This completes the proof of Proposition 3.21.

3.3.6 Asymptotically deterministic droplet growth

We now return to the analysis of the beginning of the evolution, where the droplet growth starts. Over time periods which are large but remain much smaller than $\frac{\log N}{\alpha}$ as $N \to \infty$ the total mass of type 2 is expected to grow deterministically as an exponential with rate α and constant (random) factor. The random factor developed over a short time period at the very beginning of the evolution. This leads to the idea to focus here in a first step on the case where after some time t_0 *we turn off the mutation*. The point is that the mutation which enters later than some large time t_0 is not relevant for the growth of the droplet anymore if we choose t_0 sufficiently large which we show in the second step.

In this section we therefore compute the moments of $\widehat{x}_2^N(t)$ conditioned on the configuration at a fixed time t_0, that is, given a specific realization of the random configuration $x_2^N(i,t_0),\ i = 1,\dots,N$, then identify the asymptotics as first $N \to \infty$ and then $t \to \infty$. In the Corollary 3.28 and Corollary 3.30 and then later also in the next subsubsection in Proposition 3.29 we will then consider the joint limit of $N \to \infty, t \to \infty$, for the quantity $e^{-\alpha t}\widehat{x}_2^N(t)$, more precisely $e^{-\alpha t_N}\widehat{x}_2^N(t_N)$ for $N \to \infty$ with $t_N \uparrow \infty$ and $\limsup_{N\to\infty}(t_N/\log N) < \alpha$. Those quantities converge in probability to a random limit.

In order to mimic the situation, we consider a process, where the rare mutation is turned of after time t_0 for times $t \geq 0$, we consider a process starting in finite mass and $m = 0$ for all times.

Proposition 3.27 *(Deterministic regime of droplet growth)*

Assume that at some time $t_0 \geq 0$ the distributions of $\{\widetilde{x}_2^N(t_0,i) : i = 1,\dots,N\}$ for the system (1.3), (1.4) are converging to the one of $\{\widetilde{x}_2(t_0,i), i \in \mathbb{N}\}$ with $\sum_i \widetilde{x}_2(t_0,i) < \infty$ a.s.

Moreover we assume that for every N and for the limit configuration we have that $\forall\, \varepsilon > 0$ there is a finite random set (we suppress the dependence on N in the notation):

$$\mathcal{I}(\varepsilon) = \mathcal{I}(\varepsilon, N) \text{ of } k = k(\varepsilon, N) \text{ sites,} \tag{3.837}$$

such that for every $N \geq 2$:

$$\sum_{i\in(\mathcal{I}(\varepsilon))^c} \widetilde{x}_2^N(t_0,i) < \varepsilon. \tag{3.838}$$

Assume

$$m = 0 \quad \textit{for } t \geq t_0. \tag{3.839}$$

We set (recall (3.288) for $\mathcal{U}(t,\cdot)$ and (3.285), (3.164) for $\mathcal{U}(\infty,\cdot)$):

$$g(t,x) = (1 - \sum_{\mathrm{t}}^{\infty} (1-x)^{\ell} \mathcal{U}(t,\ell)), \quad g(\infty,x) = (1 - \sum_{\ell=1}^{\infty} (1-x)^{\ell} \mathcal{U}(\infty,\ell)). \tag{3.840}$$

Then the following four convergence properties hold:

(a)

$$\begin{aligned} \lim_{t\to\infty} e^{-\alpha t} E[\beth_t^{m,t_0}([0,1])] &= \lim_{t\to\infty} e^{-\alpha t} \lim_{N\to\infty} E[\widehat{x}_2^N(t)] \\ &= E[W] \sum_{i=1}^{\infty} g(\infty, \widetilde{x}_2(t_0,i)), \end{aligned} \tag{3.841}$$

(b)

$$\lim_{t\to\infty} \lim_{N\to\infty} (E[e^{-\alpha(t+s)} \widehat{x}_2^N(t+s)] - E[e^{-\alpha t} \widehat{x}_2^N(t)]) = 0, \quad \forall s \in \mathbb{R}, \tag{3.842}$$

(c)

$$\begin{aligned} &\lim_{t\to\infty} \lim_{N\to\infty} e^{-\alpha(s+2t)} \mathrm{Cov}(\widehat{\mathrm{x}}_2^{\mathrm{N}}(\mathrm{t+s}), \widehat{\mathrm{x}}_2^{\mathrm{N}}(\mathrm{t})) \\ &\qquad = (E[W])^2 \left[2\kappa_2 \sum_{i=1}^{k} g(\infty, \widetilde{x}(t_0,i)) \right], \quad \forall s \in \mathbb{R}, \end{aligned} \tag{3.843}$$

and

(d)

$$\lim_{t\to\infty} \lim_{N\to\infty} E\left[\left([e^{-\alpha(t+s)} \widehat{x}_2^N(t+s)] - e^{-\alpha t} E[\widehat{x}_2^N(t)] \right)^2 \right] = 0, \quad \forall\ s \geq 0. \tag{3.844}$$

(e) Now assume that $m > 0$ for all $t \geq t_0$ (recall (3.839). Then the conclusions of (a)-(d) remain valid. □

We next verify that we can compute the simultaneous limits $N \to \infty$, $t \to \infty$ for times $t_N(\beta,t) = (\beta/\alpha)\log N + t$, $0 < \beta < 1$.

Corollary 3.28 *(Droplets of size N^β grow deterministically in β)*

Consider again the case where $m = 0$ and the initial state is as in Proposition 3.27. Choose β with $0 < \beta < 1$ and let

$$t_N(\beta,t) = \frac{\beta(\log N)}{\alpha} + t. \tag{3.845}$$

We have

$$\lim_{t\to\infty} \lim_{N\to\infty} E\left[\left(e^{-\alpha t} \widehat{x}_2^N(t) - e^{-\alpha t_N(\beta,0)} \widehat{x}_2^N(t_N(\beta,0)) \right)^2 \right] = 0. \qquad \square \tag{3.846}$$

This result shows that the randomness in the droplet growth is generated in the very beginning at times of order $O(1)$ as $N \to \infty$. In particular we can then show that, conditioned on the total mass of type 2 at time $t_N(\beta_1,0)$, the total mass at a later time $t_N(\beta_2,0), 1 > \beta_2 > \beta_1 \geq 0$ is *deterministic* in the limit $N \to \infty$.

Remark 49 *The limiting total droplet mass dynamic as $N \to \infty$ becomes deterministic in the following sense. Assume that $\{x_2^N(i,t_0),\ i=1,\dots,N\}$ is a measure with total mass N^β and supported on a set of size aN^β for some $a \in (0,\infty)$ and $\beta \in (0,1)$. Then we can show that for $t > t_0$ we have:*

$$\frac{Var[\widehat{x}_2^N(t)]}{(E[\widehat{x}_2^N(t)])^2} \to 0 \tag{3.847}$$

as $N \to \infty$.

Remark 50 *Given a fixed initial measure $\{\widetilde{x}(t_0,i), i \in \mathbb{N}\}$ as in the statement of Proposition 3.27 we can also prove that as $t \to \infty$*

$$\lim_{N\to\infty} E[\sum_i (x_2^N(i,t))^2] = O(\lim_{N\to\infty} E[\sum_i (x_2^N(i,t))]). \tag{3.848}$$

This is analogous to the Palm picture (3.636) and implies that the mass of type 2 is clumped on a set of sites on which the mass is almost one.

Proof of Proposition 3.27. The proof proceeds by using the dual representation of moments and the multicolour particle system introduced in Subsubsection 3.3.5 to analyse the expressions.

(a) We determine the contribution to the first moment coming only from the initial mass in $\mathcal{I}(\varepsilon)$. The result then follows by letting $\varepsilon \to 0$.

Without loss of generality we use (recall $k = k(\varepsilon,N), \mathcal{I}(\varepsilon) = \mathcal{I}(\varepsilon,N)$)

$$\mathcal{I}(\varepsilon) = \{1,\cdots,k\}. \tag{3.849}$$

Note that since we are now not working with a spatially homogeneous initial measure, we must modify the dual. In particular we eliminate the rule that single particles at a site do not move. In the calculation of $E[\widehat{x}_2^N(t)]$ we compute

$$E[\widehat{x}_2^N(t)] = \sum_{j=1}^N E[x_2^N(j,t)]. \tag{3.850}$$

To compute $E[x_1^N(j,t)]$ we start the dual with one particle at site $j \in \{1,\dots,N\}$. However the number of particles at this site will return to 1 after at most a finite random time and therefore after some finite random time the single particle will jump to a randomly chosen site and site j will become empty. Therefore for large times t the value of $E[x_1^N(j,t)]$ is independent of j (even if it is in $\mathcal{I}(\varepsilon) = \{1,\dots,k\}$).

We calculate the expectation of type 1 in a site j using the dual starting with one particle at site j.

$$E[x_1^N(t,j)] = E\left[\prod_{i=1}^{K_t^N} (1-\widetilde{x}_2^N(t_0,i))^{\zeta^N(t,i)}\right]. \tag{3.851}$$

We first consider the initial configuration where we have

$$\widetilde{x}_2^N(t_0,i) = 0 \text{ for every } i \notin \mathcal{I}(\varepsilon). \tag{3.852}$$

Recall that here $U^N(t,\ell)$ is the frequency of occupied sites that have ℓ particles.

Hence noting that there will be a $Bin(K_t^N, \frac{k}{N})$ number of hits of the k-set $\mathcal{I}(\varepsilon)$ by the dual cloud and the frequency of sites in the cloud with ℓ-sites is given by $U^N(t,\ell)$ and the contribution of such a site by $(1-\widetilde{x}_2^N(t_0,i))^\ell$, we get for sites $j \notin \{1,\cdots,k\}$ that:

$$
\begin{aligned}
(3.853)\quad E[x_1^N(t,j)] &= E\Bigg[(1-\frac{k}{N})^{K_t^N} + (1-\frac{k}{N})^{K_t^N-1}\frac{K_t^N}{N}\sum_{i=1}^k\sum_{\ell=1}^\infty U^N(t,\ell)(1-\widetilde{x}_2^N(t_0,i))^\ell \\
&\quad +(1-\frac{k}{N})^{K_t^N-2}\frac{K_t^N(K_t^N-1)}{2N^2} \\
&\quad \times \sum_{i_1,\, i_2=1}^k \sum_{\ell_1,\ell_2=1}^\infty U^N(t,\ell_1)U^N(t,\ell_2)(1-\widetilde{x}_2^N(t_0,i_1))^{\ell_1}(1-\widetilde{x}_2^N(t_0,i_2))^{\ell_2}\Bigg] \\
&\quad +o(N^{-2}).
\end{aligned}
$$

Note that as $N\to\infty$, $U^N(t,\ell)\to U(t,\ell)$, with $U(t,\cdot)$ the size distribution of the CMJ process at time t.

We rewrite the formula (3.853) as follows. Let

$$
(3.854)\quad g^N(t,x) = 1-\sum_{\ell=1}^\infty (1-x)^\ell U^N(t,\ell)
$$

and

$$
(3.855)\quad g^{N,2}(t,x) = 1-\sum_{\ell=1}^\infty (1-x)^\ell (U^N(t)^* U^N(t))(\ell).
$$

Note that $g^N, g^{N,2}$ do depend as well on the initial state of the dual process via $U^N = U^{N,(1,1)}$, this dependence we suppress here in the notation.

Then:

(3.856)

$$
\begin{aligned}
&E[x_1^N(t,j)] \\
&= E\Bigg[1-\frac{K_t^N}{N}\sum_{i=1}^k g^N(t,\widetilde{x}_2^N(t_0,i)) \\
&\qquad +\frac{K_t^N(K_t^N-1)}{2N^2}\Big\{\sum\sum\nolimits_{i_1\neq i_2}^k g^N(t,\widetilde{x}_2^N(t_0,i_1))g^N(t,\widetilde{x}_2^N(t_0,i_2)) \\
&\qquad\qquad +\sum_{i=1}^k g^{N,2}(t,\widetilde{x}_2^N(t_0,i))\Big\}\Bigg] + o(N^{-2}).
\end{aligned}
$$

Note that because of the convergence of the dual particle system to the collision-free McKean-Vlasov dual we have:

$$
(3.857)\quad g^N(t,x) \to 1-\sum_{\ell=1}^\infty (1-x)^\ell U(t,\ell) \text{ as } N\to\infty,
$$

$$
(3.858)\quad g^{N,2}(t,x) \longrightarrow 1-\sum_{\ell=1}^\infty (1-x)^\ell U^{*2}(t,\ell) \text{ as } N\to\infty.
$$

Therefore we can conclude from equation (3.856) that for $i \in \{1, 2, \cdots, k\}$:

$$E[x_2^N(t,i)] = E\left[\frac{K_t^N}{N}\sum_{i=1}^k g^N(t, \widetilde{x}_2^N(t_0,i))\right] + o(N^{-1}). \tag{3.859}$$

Consequently we calculate:

$$E[\widehat{x}_2^N(t)] = E\left[K_t^N\sum_{i=1}^k g^N(t, \widetilde{x}_2^N(t_0,i))\right] + o(1). \tag{3.860}$$

Then by $K_t^N \to K_t$ as $N \to \infty$, (3.857) and dominated convergence we get:

$$\lim_{N\to\infty} E[\widehat{x}_2^N(t)] = E\left[K_t\sum_{i=1}^k g(t, \widetilde{x}_2(t_0,i))\right] \tag{3.861}$$

where we define $g(t,x)$ using (3.857) by

$$g(t,x) = 1 - \sum_{\ell=1}^{\infty}(1-x)^\ell \mathcal{U}(t,\ell), \tag{3.862}$$

together with the fact that $K_t, \mathcal{U}(t,\cdot)$ can be identified as the number of occupied sites and the size distribution for the McKean-Vlasov limit dual process obtained above in Subsubsection 3.2.4 (Step 2).

Finally putting everything together we have:

$$\lim_{t\to\infty}(e^{-\alpha t}\lim_{N\to\infty} E[\widehat{x}_2^N(t)]) = E[W]\sum_{i=1}^k g(\infty, \widetilde{x}(t_0,i)). \tag{3.863}$$

Finally we have to remove the restriction on the initial state being 0 outside $\mathcal{I}(\varepsilon, N)$. We note that the system where $\widetilde{x}^N(0)$ is different from zero only at $\mathcal{I}(\varepsilon)$ and the true system differ at time t at most by $Const \cdot \varepsilon e^{\alpha t}$ and hence we get letting $\varepsilon \to 0$ (in $\mathcal{I}(\varepsilon)$) finally the claim of part (a).
(b) The claim on the first moment is immediate from (a).
(c) We now consider the second moment calculation again using the dual particle system. As before in the proof of part (a) we consider first the system arising by setting the type 2 mass equal to zero outside the set $\mathcal{I}(\varepsilon)$ and then later we shall let $\varepsilon \to 0$ to obtain our claim for the general case with the very same argument as in part (a).

First note that

$$\begin{aligned} E[\widehat{x}_2^N(t+s)\widehat{x}_2^N(t)] &= E[\widehat{x}_2^N(t+s)]E[\widehat{x}_2^N(t)] + cov(\widehat{x}_2^N(t+s), \widehat{x}_2^N(t)) \\ &= E[\widehat{x}_2^N(t+s)]E[\widehat{x}_2^N(t)] + \sum_{i=1}^N Cov(x_1^N(t+s,i), (x_1^N(t,i))) \\ &\qquad + \sum_{j\neq\ell}^N Cov(x_1^N(t+s,j)), x_1^N(t,\ell)). \end{aligned} \tag{3.864}$$

Since the last term of (3.864) has $N(N-1)$ summands it is necessary to consider terms of order $O(\frac{1}{N^2})$ in the computation of $Cov(x_1^N(t+s,j)), x_1^N(t,\ell))$. Therefore the calculation is organized to identify the terms up to those of order $o(N^2)$ as $N \to \infty$, and then to calculate the contributing terms of $O(N^{-2})$.

To handle the calculation of $Cov(x_1^N(t+s,j), x_1^N(t,\ell))$ both for $j \neq \ell$ and for $j = \ell = i$ we use the dual process. We introduce as "initial condition" for the dual particle system:

(3.865) one initial dual particle at j_1 at time 0 and a second dual particle at j_2 at time s

and then consider the dual cloud resulting from these at time $t+s$.

Since we shall fix times 0 and s where the j_1, respectively j_2, cloud start evolving we abbreviate the number of sites occupied by the cloud at time u and similarly the process of the frequency distribution of sizes of sites by

(3.866) $K_u^{N,j_1,j_2} = K_u^{N,(j_1,0),(j_2,s)}$ and $U^{N,j_1,j_2} = U^{N,(j_1,0),(j_2,s)}$.

Precisely this runs as follows. There will be a $Bin(K_t^{N,j_1,j_2}, \frac{k}{N})$ number of hits of the k-set $\mathcal{I}(\varepsilon)$, hence as $N \to \infty$ we obtain:

(3.867)
$$\begin{aligned}
&E[x_1^N(t+s,j_1) \cdot x_1^N(t,j_2)] \\
&= E[\prod_{\ell=1}^{K_{t+s}^{N,j_1,j_2}} (1-\widetilde{x}_2^N(t_0,\ell))^{\zeta_\ell^{(t+s)}}] \\
&= E\Bigg[(1-\frac{k}{N})^{K_{t+s}^{N,j_1,j_2}} + \frac{K_{t+s}^{N,j_1,j_2}}{N}(1-\frac{k}{N})^{K_{t+s}^{N,j_1,j_2}-1} \sum_{i=1}^{k}\sum_{\ell=1}^{\infty} U_{t+s}^{N,j_1,j_2}(\ell)(1-\widetilde{x}_2^N(t_0,i))^{\ell} \\
&\qquad +(1-\frac{k}{N})^{K_{t+s}^{N,j_1,j_2}-2}\frac{K_{t+s}^{N,j_1,j_2}(K_t^{N,j_1,j_2}-1)}{2N^2} \\
&\qquad \sum_{i_1,\,i_2=1}^{k}\sum_{\ell_1,\ell_2=1}^{\infty} U^{N,j_1,j_2}(t+s,\ell_1)U^{N,j_1,j_2}(t+s,\ell_2) \\
&(1-\widetilde{x}_2^N(t_0,i_1))^{\ell_1}(1-\widetilde{x}_2^N(t_0,i_2))^{\ell_2}\Bigg] + o(N^{-2}).
\end{aligned}$$

Therefore if we now use the function $g^N(t,x)$ defined in (3.854) but now for the initial state specified by the space-time points $(j_1,0),(j_2,s)$ (note $g^N = g^{N,(j_1,0)}$, respectively $= g^{N,(j_2,0)}$), we can rewrite the above expression. However there is a small problem with the time index, since our two clouds now have a time delay s therefore depending on which of the two clouds has the hit of $\mathcal{I}(\varepsilon)$ we have to use t or $t+s$ as the argument in our function $g^N(\cdot,x)$. Here we use therefore the *notation* $t(\cdot)$ which indicates that actually the time parameter here is either t or $t+s$ and depends on which of the two clouds is involved. However this abuse of notation is harmless since in the arguments below both the times t and $t+s$ will go to ∞.

We get, with the above conventions, from the formula (3.867) that:

(3.868)
$$\begin{aligned}
&E[x_1^N(t+s,j_1)x_1^N(t,j_2)] \\
&= 1 - E\left[\frac{K_{t+s}^{N,j_1,j_2}}{N}\sum_{i=1}^{k} g^N(t(\cdot),\widetilde{x}_2^N(t_0,i))\right] \\
&+E\left[\frac{K_{t+s}^{N,j_1,j_2}(K_{t+s}^{N,j_1,j_2}-1)}{2N^2}\sum_{i_1\neq i_2=1}^{k} g^N(t(\cdot),\widetilde{x}_2^N(t_0,i_1))g^N(t(\cdot),\widetilde{x}_2^N(t_0,i_2))\right] + o(N^{-2}).
\end{aligned}$$

Now we continue and calculate the covariances of $x_1^N(t+s,j_1)$ and $x^N(t,j_2)$ based on (3.868) and the first moment calculations. For this purpose we can consider the dual systems starting in $(j_1,0)$ respectively (j_2,s) and evolving *independently* and the one starting with a factor at each time-space point $(j_1,0)$ and (j_2,s) but evolving together and with *interaction*. As we saw we can couple all these evolutions via a multicolour particles dynamic. This now allows to calculate with the first moment formulas obtained in (3.851) to (3.862) and using the notation

$$s_m = 0 \text{ if } m=1 \text{ and } s_m = s \text{ if } m=2, \tag{3.869}$$

and we obtain the covariance formula

$$\begin{aligned}
&\mathrm{Cov}(x_1^N(t+s,j_1)x_1^N(t,j_2)) \\
&= 1 - E\left[\frac{K_{t+s}^{N,j_1,j_2}}{N}\sum_{i=1}^k g^N(t(\cdot),\widetilde{x}_2^N(0,i))\right] \\
&\quad + E\left[\frac{K_{t+s}^{N,j_1,j_2}(K_{t+s}^{N,j_1,j_2}-1)}{2N^2}\sum_{i_1\neq 1_2=1}^k g^N(t(\cdot),\widetilde{x}_2^N(t_0,i_1))g^N(t(\cdot),\widetilde{x}_2^N(0,i_2))\right] \\
&\quad - \prod_{m=1}^2 E\Bigg[1-\frac{K^{N,j_m}(t+s_m)}{N}\sum_{i_m=1}^k g^N(t+s_m,\widetilde{x}_2^N(t_0,i_m)) \\
&\quad + \frac{K_{t+s_m}^{N,j_m}(K_{t+s_m}^{N,j_m}-1)}{2N^2}\sum_{i_{m,1}=1}^k\sum_{i_{m,2}=1}^k g^N(t+s_m,x_{i_{m,1}})g(t+s_m,x_{i_{m,2}})\Bigg] \\
&\quad + o(N^{-2}).
\end{aligned} \tag{3.870}$$

We now have to distinguish two cases, namely $j_1\neq j_2$ and $j_1=j_2$ and we argue separately in the two cases.

From (3.870), using (3.859) and Proposition 3.24, we get that if $j_1\neq j_2$, then

$$\begin{aligned}
&\mathrm{Cov}(x_2^N(t+s,j_1),x_2^N(t,j_2)) \\
&= \frac{1}{N^2}\left\{E[N(K_{t+s}^{N,j_1}+K_t^{N,j_2}-K_{t+s,t}^{N,j_1,j_2})\sum_{i=1}^k g^N(t(\cdot),\widetilde{x}_2^N(t_0,i))]\right\} + o(N^{-2}).
\end{aligned} \tag{3.871}$$

The expression for $j_1=j_2$ is a slight modification of this, since either we have as first step a coalescence and are left with one particle or we get first a migration event and get a $j_1\neq j_2$ case, or finally we get a birth event first. Since we have the time delay s of course coalescence can act only with this delay.

Therefore combining both cases up to terms of order $o(\frac{1}{N^2})$ as $N\to\infty$,

$$\begin{aligned}
&\mathrm{Cov}(\widehat{x}_2^N(t+s),\widehat{x}_2^N(t)) \\
&= \left\{E[N(K_{t+s}^{N,j_1}+K_t^{N,j_2}-K_{t+s,t}^{N,j_1,j_2})\sum_{i=1}^k g^N(t,\widetilde{x}_2^N(t_0,i))]\right\} + o(1).
\end{aligned} \tag{3.872}$$

Recall that $g^N(t,\widetilde{x}_2^N(t_0,i_1))\to g(t,\widetilde{x}_2(t_0,i_1))$ as $N\to\infty$. Therefore the main point is to identify the behaviour of

$$N(K_{t+s}^{N,j_1}+K_{t+s}^{N,j_2}-K_{t+s}^{N,j_1,j_2}),\ \text{as } N\to\infty, \tag{3.873}$$

which is N times the difference between two independent dual populations each starting in j_1 and j_2 and one combined interacting system. In order to evaluate this we must determine the effect of the collisions of the dual populations.

In order to analyse the dual representation of the r.h.s. of equation (3.864) rewritten in the form (3.872) we will again work with the enriched coloured particle system which allows us to identify the contribution, on the one hand of the two evolving independent clouds, and on the other hand the effects of the interaction between the two clouds by collision and subsequent coalescence.

The construction of the coloured particle system (observing $W_1, W_2, R_1, R_2, R_{1,2}, G_1, G_2, G_{1,2}$ particles which are relevant for the accuracy needed here) was given above in the sequel of (3.759). Recall that the $G_{1,2}$ particles are produced in the coloured particle system from collision of W_1 and W_2 particles, that is, W_1 and W_2 particles at the same site which then suffer coalescence. Such an event reduces the number of occupied sites in the interacting clouds by 1 and then creates a $G_{1,2}$ family. Then the number of occupied sites in the W_1, W_2 interacting system K_t^{N,j_1,j_2} is given by (3.742).

First we observe that if we evaluate the duality relation (recall the form of the initial state we consider here) we see that the contributing terms of order $O(\frac{1}{N^2})$ arise from either double or single hits of the dual particle cloud of a point in $\mathcal{I}(\varepsilon)$.

We note that in general there are two cases which must be handled differently, namely, the case in which the difference in (3.873) is based on a Poisson collision event with mean of order $O(\frac{1}{N})$ as in Lemma 3.26 (b) and the other in which there is a law of large numbers effect as in(3.485) and Lemma 3.26(a). However in both cases the expected value used in the calculations below have the same form. In the present case for fixed t we are in the Poisson case with vanishing rate so that at most one Poisson event contributes to the limit.

Using Proposition 3.24 (3.742) we se that the key quantity in (3.873) satisfies as $N \to \infty$ that:

$$E[N(K_{t+s}^{N,j_1} + K_t^{N,j_2} - K_{t+s,t}^{N,j_1,j_2})] \sim N \cdot E[\mathfrak{g}_{(1,2)}(W'(\cdot), W''(\cdot), N, s, t)] \tag{3.874}$$

where $\mathfrak{g}_{(1,2)}(W'(\cdot), W''(\cdot), N, s, t)$ is given by (3.735) and has mean given in (3.745). Hence we conclude with (3.745) that the limit of the r.h.s. of (3.874) is given by

$$2\kappa_2 (E[W])^2 e^{\alpha(2t+s)}. \tag{3.875}$$

Remark 51 *Here we have noted that using the coupling introduced above the number of* single hits *of the k sites in $\mathcal{I}(\varepsilon)$ coming from either one of the two interacting clouds, can be obtained by deducting from the single hits of the "non-interacting W_1 and W_2 particles" the single hits by $G_{1,2}$ particles in the interacting system. Moreover the event of a* double hit, *that two occupied sites in the dual cloud coincide with one of the $k = k(\varepsilon)$ sites, can occur from either two W_1 sites, two W_2 sites or one W_1 and one W_2 site. (Since the number of $G_{1,2}$ sites or including sites at which both W_1 and W_2 particles coexist are of order $O(\frac{1}{N})$, the contribution of double hits involving $G_{1,2}$ sites is of higher order and can be ignored.)*

Taking the limit as $N \to \infty$ in (3.872), using (3.875), that leads to

$$\begin{aligned} &Cov(\widehat{x}_2^N(t+s), \widehat{x}_2^N(t)) \underset{N\to\infty}{\longrightarrow} \\ &2\kappa_2 e^{\alpha(2t+s)} E[(W'(t+s)W''(t) \sum_{i=1}^k g(t, \widetilde{x}_2(t_0, i)]. \end{aligned} \tag{3.876}$$

Recalling (3.862), (3.757) and the fact that $\mathcal{U}(t,k) \to \mathcal{U}(\infty, k)$ as $t \to \infty$, we get next as $t \to \infty$:

$$\begin{aligned} &\text{r.h.s. of (3.876)} \sim 2e^{\alpha(2t+s)} E\left[\kappa_2 W' W'' \textstyle\sum_{i=1}^k g(t, \widetilde{x}_2(t_0, i))\right] \text{ and} \\ &\left| e^{-(2t+s)} Cov(\widehat{x}_2^N(t+s), \widehat{x}_2^N(t)) - 2(E[W])^2 \left[\kappa_2 \sum_{i=1}^k g(\infty, \widetilde{x}_2(t_0, i))\right]\right| \to 0, \end{aligned} \tag{3.877}$$

uniformly in s. To verify the uniformity we work with the expression for the expected number of $G_{1,2}$ sites (in the $N \to \infty$ limit dynamics), recall here (3.753), to get:

$$E\left[\int_s^{t+s} W'(u)W''(u-s)e^{\alpha(u-s)}e^{\alpha u}W^g(t+s,u)e^{\alpha(t+s-u)}du\right]. \tag{3.878}$$

Then using that for the $N \to \infty$ limiting dynamics always $W(u) \to W$ as $u \to \infty$ and showing that the contribution from the early collisions (in $[0, s+s_0(t)]$, $s_0(t) = o(t)$) form a negligible contribution as in the argument following (3.757) we obtain

$$\begin{aligned} e^{-\alpha(2t+s)}E&\left[\int_s^{t+s} W_1(u)W_2(u-s)e^{\alpha(u-s)}e^{\alpha u}W^g(t+s,u)e^{\alpha(t+s-u)}du\right] \\ &-(E[W])^2\left[\kappa_2\sum_{i=1}^k g(\infty,\widetilde{x}_2(t_0,i))\right] \leq \mathrm{const}\cdot \mathrm{e}^{-\alpha \mathrm{t}} \text{ as } \mathrm{t}\to\infty. \end{aligned} \tag{3.879}$$

This proves (3.843) and completes the proof of (c).

(d) follows immediately from (c) and (a).

(e) To verify this note let $\widehat{x}_2^{N,m,[t_0,\infty)}(t)$ denote the contribution to the population resulting from rare mutations that occur after t_0. Then

$$E[e^{-\alpha t}\lim_{N\to\infty}\widehat{x}_2^{N,m,[t_0,\infty)}(t)] \leq \frac{m}{\alpha}e^{-\alpha t_0}. \tag{3.880}$$

Since the above results are valid for any t_0, we can let $t_0 \to \infty$, to conclude that they remain true if we do not turn off the rare mutation at time t_0.

Remark 52 *In the above calculations we have included double hits - that is two clouds including the same site and that conditioned on a hit the number of particles at that site is given by the size distribution and therefore converges to the stable size distribution as $t \to \infty$.*

Proof of Corollary 3.28

Again we use the dual representation of the moments and analyse the dual based on the enriched coloured particle system. We consider separately two cases $0 < \beta < 1/2$ and $1/2 \leq \beta < 1$, which correspond to the distinction whether two clouds of dual particles descending from the two initial particles do not meet as $N \to \infty$ in the $(\beta/\alpha)\log N$ time scale with a nontrivial probability or whether they do next.

The proof in the case $0 < \beta < 1/2$ (where the clouds do not meet asymptotically) follows the same lines as for Proposition 3.27 only that now in (3.829) the limits $N \to \infty$ and $s \to \infty$ are carried out at once. Hence the additional step is to verify that the error terms remain negligible when s is replaced by s_N with

$$s_N = t_N(\beta) - t. \tag{3.881}$$

Now we have to analyse the first and second moments in this time scale.

To prove the claims we first note that for $\beta < 1$ the probability that the particle system hits a given site still goes to 0 as $N \to \infty$. Therefore with (3.860) we still can apply the CMJ-theorem on exponential growth which takes care of the *first moment* term asymptotic.

Moreover, the remainder term in the calculation of the *second order term s*, i.e. of

$$E[e^{-\alpha t}x_1^N(t,j_1)e^{-\alpha t_N(\beta)}x_1^N(t_N(\beta),j_2)] \tag{3.882}$$

has the order

$$O(\frac{e^{\alpha t_N(\beta)}}{N^3}) = o(N^{-2}). \tag{3.883}$$

This gives the claim.

The additional consideration in case $\frac{1}{2} < \beta < 1$ is that there is a positive probability that two clusters will collide in this time scale. However the question is whether the collisions with their effects can change the occupation integral appearing in the dual representation in the exponential. We see that since the number of those sites at which collisions occur among the total of the N sites is $o(N)$ we do not need to consider higher level occupation.

3.4 Relation between ${}^*\mathcal{W}$ and $\mathcal{W}^*$

Let us briefly review the two limiting regimes we have considered. In the first regime we have considered the limiting droplet process $\daleth^m_t$ which is a $\mathcal{M}_a([0,1])$-valued Markov process. We have also proved that there exists $\alpha^* > 0$ such that

$$\{e^{-\alpha^* t}\daleth^m_t([0,1])\}_{t\geq 0} \text{ is tight} \tag{3.884}$$

and determined the limiting first moment and variance of $e^{-\alpha^* t}\daleth^m_t([0,1])$ as $t \to \infty$. This has been obtained by considering the process $\daleth^{N,m}_t$ for times in $[0, t_N]$ with $t_N \to \infty,\ t_N - \frac{\log N}{\alpha^*} \to -\infty$ and letting $N \to \infty$. Moreover by (3.844) it follows that $\daleth^m_t([0,1])$ is Cauchy as $t \to \infty$ and therefore has a limit $\mathcal{W}^*$ in L^2 as $t \to \infty$.

On the other hand there exists $\alpha > 0$ such that the empirical process $\int_0^1 x\Xi^{\log,\alpha}_N(t, dx)$ in time window $\{\frac{\log N}{\alpha} + t\}_{t\in\mathbb{R}}$ converges to $\mathcal{L}_t$, and the limit ${}^*\mathcal{W} = \lim_{t\to-\infty} e^{-\alpha t}\int_0^1 x\mathcal{L}_t(dx)$ exists and is random.

Next recall that we have established that the two exponential growth rates α and α^* in the two time regimes $t_N \uparrow \infty, t_N = o(\log N)$ as $N \to \infty$ respectively $t_N = \frac{1}{\alpha}\log N + t$ as $N \to \infty, t \to -\infty$ are the same, so that in that respect limits interchange.

It is then natural to conjecture that also the distributions of ${}^*\mathcal{W}$ and $\mathcal{W}^*$ are the same even though there is the following obstacle. Define

$$t_N(\beta, t) = \frac{\beta}{\alpha}\log N + t, \text{ with } 0 < \beta < 1. \tag{3.885}$$

Consider the law $\mathcal{L}[\bar{x}^N_2(t_N(\beta,t))]$ and note that for fixed t there is a discontinuity at $\beta = 1$ since then collisions of the dual occur at order $O(1)$ and we know that as function of β we jump from type-2 mass zero for $\beta < 1$ to a positive value at $\beta = 1$. Hence the matter to relate ${}^*\mathcal{W}$ and $\mathcal{W}^*$ is very subtle.

In order to investigate the relation between the laws of ${}^*\mathcal{W}$ and $\mathcal{W}^*$ we next consider the relations between the dual process at times corresponding to the droplet formation and the dual in the time scale corresponding to the macroscopic emergence. In this direction we verify first that the first two moments of $\mathcal{W}^*$ and ${}^*\mathcal{W}$ agree and then use the techniques of proof to conclude with L_2-arguments that the two random variables have asymptotically as $N \to \infty, t \to +\infty$ respectively $N \to \infty, t \to -\infty$ L_2-distance zero.

Recall the notation $t_N(\beta,t) = \beta\alpha^{-1}\log N + t \quad , \quad t_N = \alpha^{-1}\log N,$ which we shall use in the sequel.

Proposition 3.29 *(Relation between ${}^*\mathcal{W}$ and $\mathcal{W}^*$)*

(a) For $0 < \beta < 1$,

$$\lim_{t\to-\infty}\lim_{N\to\infty} E\left[\left(e^{-\alpha t_N(\beta,0)}\widehat{x}^N_2(t_N(\beta,0)) - e^{-\alpha(\frac{\log N}{\alpha}+t)}\widehat{x}^N_2(\frac{\log N}{\alpha}+t))\right)^2\right] = 0. \tag{3.886}$$

(b) We have the relation:

$$\mathcal{L}[{}^*\mathcal{W}] = \mathcal{L}[\mathcal{W}^*]. \tag{3.887}$$

□

Remark 53 *We can summarize the results in (3.886) as follows. Consider for* $0 < \beta < 1$, *with* $\bar{x}_2^N(t) := \frac{\widehat{x}_2^N(t)}{N}$, *the quantities:*

$$(3.888) \quad \{\widehat{x}_2^N(t) : 0 \le t \le \frac{\beta \log N}{\alpha}\}, \quad \{\bar{x}_2^N(\frac{\log N}{\alpha} + t) : \frac{(\beta-1)\log N}{\alpha} \le t \le 0\}.$$

Then we have shown so far that these two processes converge in law to

$$(3.889) \quad \{\widehat{x}_2(t) : 0 \le t < \infty\}, \quad \{\bar{x}_2(t) : -\infty < t \le 0\}, \text{ respectively.}$$

We can, by the classical embedding theorem of weakly convergent distributions on a Polish space, construct both sequences and their limits on one probability space $(\Omega, \mathcal{A}, P)$ *preserving the law of each random variable in question and such that we have a.s. convergence in the supnorm on compact time intervals. On this probability space we consider* $L^2(\Omega, \mathcal{A}, P)$.

Then we have:

$$(3.890) \quad \mathcal{W}^* = L^2 - \lim_{t\to\infty} e^{-\alpha t}\widehat{x}_2(t) = L^2 - \lim_{t\to-\infty} e^{-\alpha t}\bar{x}_2(t) =^* \mathcal{W}.$$

Corollary 3.30 *(Equality of moments for growth constant)*

$$(3.891) \quad E[(^*\mathcal{W})^k] = E[(\mathcal{W}^*)^k], \quad k = 1, 2, \dots. \qquad \square$$

Proof of Corollary 3.30 This follows from Proposition 3.29.

Remark 54 *We can calculate the moments of* $^*\mathcal{W}$ *and* $\mathcal{W}^*$ *by explicit calculation. However since we have not proved that all moments are finite we cannot use this to prove the equality in law (3.887)).*

Proof of Proposition 3.29

We start with the rough idea. We have calculated $Var[\mathcal{W}^*]$ in (3.835) which involved the time scales smaller than $\alpha^{-1}\log N$. An expression for $Var[^*\mathcal{W}]$ was derived in (3.617). We see that $Var[\mathcal{W}^*] = Var[^*\mathcal{W}]$. We shall now prove (3.886) which will give (3.887). The point is that the moment calculations now have to include time scales up to $\alpha^{-1}\log N$ and must therefore different from before treat collisions of the dual cloud for a given point occuring with positive probability.

We proceed in five steps. We again start by considering the case without mutation but start instead with positive but finite initial type-2 mass. We calculate in Step 1 first moments, in Step 2 second moments at the given time, in Step 3 we calculate the mixed moment between the two times specified in the assertion under our assumption. In Step 4 of our arguments we include mutation. In Step 5 we conclude the proof.

Step 1 We begin with the *first moment* calculation. Consider the calculation of the expected value using the dual particle system starting with one particle at j and recall $\mathcal{I}(\varepsilon)$ from (3.849). We will assume that at time 0 we have an initial distribution as in Proposition 3.27 and that furthermore $m = 0$. As above there will then be a $\text{Bin}(K^N_{t_N(1,t)}, \frac{k}{N})$ distributed number of hits of the k-set $\mathcal{I}(\varepsilon)$ by the cloud of dual particles. To apply the Poisson approximation recall that by (3.738)-(3.741) we have:

$$(3.892) \quad K_t^{N,(a)} = W_a(t)e^{\alpha t} - \mathfrak{g}(W_a(\cdot), N, t) + \mathcal{E}_a(N, t),$$

where the error term satisfies

$$(3.893) \quad E[\mathcal{E}_a(N,t)] \le \mathrm{const}\frac{\mathrm{e}^{2\alpha t}}{\mathrm{N}} \cdot \sup_{\mathrm{u}} \mathrm{E}[\mathrm{W_g(u)}].$$

Therefore the number of hits of the k-set converges as $N \to \infty$ weakly to a *Poisson distribution with parameter* $We^{\alpha t} - \kappa_2 W^2 e^{2\alpha t} + O(e^{3\alpha t})$. Therefore from (3.867) we get looking at 0,1 or 2 hits of $\mathcal{I}(\varepsilon)$ and denoting by

$$\{\xi_\ell, \quad \ell = 1, \cdots, K^N_{t_N(1,t)}\} \tag{3.894}$$

the states of the sites occupied by the dual at time $t_N(1,t)$ and by K^N_t, U^N_t the number of occupied sites and the size distribution for the finite N dual (nonlinear) system studied above:

$$\begin{aligned} E[x_1^N(t_N(1,t),j)] &= E\left[\prod_{\ell=1}^{K^N_{t_N(1,t)}} (1 - x_2^N(t_0,\xi_\ell))^{\zeta^N(t_N(1,t),\xi_\ell)}\right] \\ &= E\Bigg[(1-\frac{k}{N})^{K^N_{t_N(1,t)}} \\ &+(1-\frac{k}{N})^{K^N_{t_N(1,t)}-1} \quad \frac{K^N_{t_N(1,t)}}{N} \sum_{i=1}^k \sum_{\ell=1}^\infty U^N(t_N(1,t),\ell)(1-\widetilde{x}_2^N(t_0,i))^\ell \\ &+(1-\frac{k}{N})^{K_{t_N(1,t)}-2} \quad \frac{K^N_{t_N(1,t)}(K^N_{t_N(1,t)}-1)}{2N^2} \\ &\sum_{i_1,\, i_2=1}^k \sum_{\ell_1,\ell_2=1}^\infty U^N(t_N(1,t),\ell_1)U^N(t_N(1,t),\ell_2)(1-\widetilde{x}_2^N(t_0,i_1))^{\ell_1}(1-\widetilde{x}_2^N(t_0,i_2))^{\ell_2}\Bigg] \\ &+o(N^{-2}). \end{aligned} \tag{3.895}$$

We rewrite this in the form

$$\begin{aligned} &E[x_1^N(t_N(1,t),j)] \\ &= E\Bigg[1 - (W_N(t_N)e^{\alpha t} + O(W_N^2(t_N)e^{2\alpha t})) \sum_{i=1}^k g^N(t_N(1,t),\widetilde{x}_2^N(t_0,i)) \\ &\qquad + \frac{W_N(t_N)e^{2\alpha t}}{2} \sum_{i_1=1}^k \sum_{i_2=1}^k g^N(t_N(1,t),x_{i_1}) g^N(t_N(1,t),x_{i_2})\Bigg] + o(N^{-2}), \end{aligned} \tag{3.896}$$

with

$$g^N(t_N(1,t),x) = 1 - \sum_{\ell=1}^\infty (1-x)^\ell U^N(t_N(1,t),\ell). \tag{3.897}$$

Therefore as $N \to \infty$, we have (recall that $W_N(t_N) \to W$ a.s. as $N \to \infty$ and $W_N(t_N)$ are uniformly square integrable):

$$E[x_2^N(t_N(1,t),j)] = E\left[(We^{\alpha t} + O(W^2e^{2\alpha t})) \sum_{i=1}^k g^N(t_N(1,t),\widetilde{x}_2^N(t_0,i))\right] + o(N^{-2}). \tag{3.898}$$

Next we need to know more about $g^N(t_N(1,t),\widetilde{x}_2^N(t_0,i))$ as $N \to \infty$ and then $t \to -\infty$. Recall that $g^N(t_N(1,t),x) = 1 - \sum_{\ell=1}^\infty (1-x)^\ell U^N(t,\ell)$ (see 3.854), $\lim_{N\to\infty} U^N(t_N(1,t),\cdot) = U(t,\cdot)$ (see 3.51) and $\lim_{t\to-\infty} U(t,\cdot,\cdot) = \mathcal{U}(\infty,\cdot,\cdot)$ (see 3.362). Therefore

$$\lim_{t\to-\infty} \lim_{N\to\infty} g^N(t_N(1,t),\widetilde{x}_2^N(t_0,i)) = g(\infty,\widetilde{x}_2(t_0,i)) = 1 - \sum_{\ell=1}^\infty (1-\widetilde{x}_2(t_0,i))^\ell \mathcal{U}(\infty,\ell). \tag{3.899}$$

This allows to calculate next as $N \to \infty, t \to -\infty$:

$$E[\widehat{x}_2^N(t_N(1,t))] = E\left[WNe^{\alpha t}\sum_{i=1}^{k} g^N(t_N(1,t),\widetilde{x}_2^N(t_0,i))\right] + o(e^{\alpha t}N^{-1}). \tag{3.900}$$

Then

$$\lim_{t\to-\infty}\lim_{N\to\infty} E[e^{-\alpha t_N(1,t)}\widehat{x}_2^N(t_N(1,t))] = E\left[W\sum_{i=1}^{k} g(\infty,\widetilde{x}_2(t_0,i))\right]. \tag{3.901}$$

Finally putting everything together results in

$$\lim_{t\to-\infty}\lim_{N\to\infty} E[e^{-\alpha t_N(1,t)}(\widehat{x}_2^N(t_N(1,t))] = E[W]\sum_{i=1}^{k} g(\infty,\widetilde{x}(t_0,i)). \tag{3.902}$$

Step 2 We now consider the *second moments*. Consider the time points $t_N(\beta_1,0)$ and $t_N(\beta_2,t), \beta_1 < \beta_2$. The case $\beta < 2$ follows as in Proposition 3.27. The case $\beta = 1$ requires more careful analysis, we carry out now. Let $0 < \beta \leq 1$, then

$$\begin{aligned}
&E[x_1^N(t_N(\beta,0),j_1)\cdot x_1^N(t_N(1,t),j_2)] = \\
&= E[\prod_{j=1}^{K^{N,j_1,j_2}_{t_N(1,t)}}\prod_{\ell=1}^{\zeta(t_N(1,t))}(1-\widetilde{x}_2^N(t_0,j))^{\ell}] \\
&= E\Bigg[(1-\frac{k}{N})^{K^{N,j_1,j_2}_{t_N(1,t)}} + \frac{K^{N,j_1,j_2}_{t_N(1,t)}}{N}(1-\frac{k}{N})^{K^{N,j_1,j_2}_{t_N(1,t)}-1}\sum_{i=1}^{k}\sum_{\ell=1}^{\infty}U_t^N(\ell)(1-\widetilde{x}_2^N(t_0,i))^{\ell} \\
&+(1-\frac{k}{N})^{K^{N,j_1,j_2}_{t_N(1,t)}-2}\frac{K^{N,j_1,j_2}_{t_N(1,t)}(K^{N,j_1,j_2}_{t_N(1,t)}-1)}{2N^2} \\
&\sum_{i_1,\,i_2=1}^{k}\sum_{\ell_1,\ell_2=1}^{\infty}U^N(t,\ell_1)U^N(t,\ell_2)(1-\widetilde{x}_2^N(t_0,i_1))^{\ell_1}(1-\widetilde{x}_2^N(t_0,i_2))^{\ell_2}\Bigg] \\
&+o(N^2).
\end{aligned} \tag{3.903}$$

Proceeding as in the proof of Proposition 3.27 in the case where $j_1 \neq j_2$ we get:

$$\begin{aligned}
&\mathrm{Cov}(x_2^N(t_N(\beta,0),j_1),x_2^N(t_N(1,t),j_2)) \\
&= \frac{1}{N^2}\left\{E\left[N\left(K^{N,j_1}_{t_N(1,t)} + K^{N,j_2}_{t_N(\beta,0)} - K^{N,j_1,j_2}_{t_N(1,t),t_N(\beta,0)}\right)\sum_{i=1}^{k} g^N(t_N(\cdot),\widetilde{x}_2^N(t_0,i))\right]\right\} \\
&+o(N^{-2}).
\end{aligned} \tag{3.904}$$

Step 3 The next point is to prove the following claim on second moments for $N \to \infty$ and then as $t \to -\infty$. For $0 < \beta \leq \beta_2 \leq 1$,

$$\lim_{t\to-\infty}\lim_{N\to\infty} E\left[\left(e^{-\alpha t_N(\beta,0)}\widehat{x}_2^N(t_N(\beta,0))\cdot e^{-\alpha t_N(\beta_2,t)}\widehat{x}_2^N(t_N(\beta_2,t)))\right)\right] = \text{const.}, \tag{3.905}$$

with a constant independent of β, β_2.

In order to verify (3.905) we must now include new considerations that arise since $e^{\alpha t_N(1,t)} = Ne^{\alpha t}$ leading to a positive particle density and hence collisions. Recall our representation in terms

of the *coloured particle system* and a careful analysis of the number of W_1, W_2 and finally $G_{1,2}$ particles. We begin with the W_1 and W_2 particles.

The key point is that in the analysis of the κ_2 term one needs to include slower growth of the W_1 and W_2 particles due to the nonlinear term. In particular the populations of white particles are asymptotically as $N \to \infty$ of the form $NF(t)$ and the function $F(t)$ can be expanded as $t \to -\infty$ in terms of k-th powers $(k = 1, 2, \cdots)$ of $e^{\alpha t}$. Hence the number of particles in the two white clouds, the one starting in j_1 and the one starting in j_2 are each asymptotically as $N \to \infty$ behaving as

$$We^{\alpha t_N(\gamma,t)} - \frac{1}{N}\kappa_2 W^2 e^{2\alpha t_N(\gamma,t)} \sim WNe^{\alpha t} - \kappa_2 W^2 N e^{2\alpha t} = Ne^{\alpha t} W(1 - \kappa_2 W e^{\alpha t}), \tag{3.906}$$

where we take two independent realisations of W for each cloud and with $\gamma = 1 - \beta$. The system with start of a particle at j_1 at time 0 and j_2 at time $t_N(\beta, 0)$ evolves similar but now an additional correction is needed because of the collisions between the two clouds. This effect is represented by the $G_{1,2}$ particles.

Hence the important term for the covariance calculation is the one involving the $G_{1,2}$ particles. The number of $G_{1,2}$ particles is taking again $N \to \infty$ and then afterwards $t \to -\infty$ of order

$$O(W'W''\kappa_2 e^{2\alpha t}). \tag{3.907}$$

The self-intersections of the W_1 and the $G_{1,2}$ particles produce only a higher order correction to the number and size distribution of the $G_{1,2}$ particles. As a result in the limit as $t \to -\infty$ the correction terms are all of higher order.

From together with (3.904, 3.906) and (3.907) we can verify that the covariance term is asymptotically the same as (3.843), that is,

$$\begin{aligned}
&\lim_{t\to-\infty} \lim_{N\to\infty} \operatorname{cov}(e^{-\alpha(t_N(1,t))}\widehat{x}_2^N(t_N(1,t)), e^{-\alpha t_N(\beta,0)}\widehat{x}_2^N(t_N(\beta,0))) \\
&= 2(E[W])^2 \left\{ \kappa_2 \sum_{i=1}^{k} g(\infty, \widetilde{x}_2(t_0, i)) \right\}.
\end{aligned} \tag{3.908}$$

and we get the claim in (3.905).

Step 4 We are now ready to conclude the argument for the two assertions of the proposition. First we have to show the initial masses we used in the calculation above can in fact arise as configuration at time t_0. This follows from (1.76), where we show convergence of the $\daleth^{m,N}$ to $\daleth^m$ (recall (1.59)). Second we have to prove that the case we treated can be replaced by the masses which arise from mutation. For any fixed t_0 these masses satisfy (3.837), (3.838) and the above argument gives therefore the result for mutation turned of after time t_0. We now have to argue that this is not relevant for large t_0.

The idea is that the contribution to $\mathcal{W}^*$ due to mutation after t_0 is negligible for large t_0, more precisely, the expected contribution is $O(e^{-\alpha t_0}) \to 0$ as $t_0 \to \infty$. The result then follows by letting $t_0 \to \infty$ in Corollary 3.31 below.

Step 5 Next we have to go through the list of claims.

(a) This follows from (3.908) above, (3.846) together with (3.843) and (3.905) for $\beta = \beta_2 = 1$ since we see that both the variances of the two random variables and their covariance all converge to the same value.

(b) Follows from (a) by using the construction of Remark 53 ($\mathcal{W}^*, {}^*\mathcal{W}$) both on one probability space) and noting that with $t_N(0, t) = t$

$$e^{-\alpha t_N(1,t)}\widehat{x}_2^N(t_N(1,t)) = e^{-t}\bar{x}_2^N(t_N(1,t)), \tag{3.909}$$

$$\mathcal{W}^* = \lim_{t\to\infty} \lim_{N\to\infty} e^{-\alpha t_N(0,t)}\widehat{x}_2^N(t_N(0,t)) \tag{3.910}$$

and

$$(3.911) \quad {}^*\mathcal{W} = \lim_{t\to-\infty}\lim_{N\to\infty} e^{-t}\bar{x}_2^N(t_N(1,t)).$$

This completes the proof of Proposition 3.29 if we could show that in (3.911) we can replace $t_N(0,t)$ and $t \to \infty$ by t_N with $t_N \to \infty$ but $t_N = o(\log N)$. Hence the proof of Proposition 3.29 follows from the next corollary.

Corollary 3.31 *Consider $t_N \uparrow \infty$ with $t_N = o(\log N)$. Then:*

$$(3.912) \quad \lim_{t\to\infty}\lim_{N\to\infty} E[(e^{-\alpha t}\widehat{x}_2^N(t) - e^{-\alpha t_N}\widehat{x}_2^N(t_N))^2] = 0 \qquad \square$$

Proof of Corollary 3.31

If we consider the case in which we exclude mutation but insert initially some finite mass (even for $N \to \infty$) then the result follows due to (3.846). What we want is however the statement where initially we have no type-2 mass but where we have mutation from type 1 to type 2 at rate mN^{-1}. The idea now is to use the fact that only early mutation counts, that is, mutation after some late time has a negligible effect. We can use this idea together with coupling to prove the result with mutation.

We consider solutions of our basic equation for mutation rate m and mutation rate 0. We denote the solutions by

$$(3.913) \quad (X_t^{N,m})_{t\geq 0} \text{ and } (X_t^{N,0})_{t\geq 0}.$$

We can define these two processes on one probability space so that the following relation holds:

$$(3.914) \quad \widehat{x}_2^{N,0}(t) \leq \widehat{x}_2^{N,m}(t),$$

with

$$(3.915) \quad E[\widehat{x}_2^{N,m}(t) - \widehat{x}_2^{N,0}(t)] \leq Const \cdot m \int_0^t e^{\alpha s} ds.$$

We now apply this to compare the process with mutation turned off after time t_0 with the original process with mutation running on $[0,\infty)$. We know $\widehat{x}_2^{N,0}(t_0) = \widehat{x}_2^{N,m}(t_0) = O(e^{\alpha t_0}))$ and with (3.914) and (3.915) we have

$$(3.916) \quad 0 \leq E[e^{-\alpha(t_0+t)}\widehat{x}_2^{N,m}(t_0+t) - e^{-\alpha(t+t_0)}\widehat{x}_2^{N,0}(t_0+t)] \leq Const \cdot me^{-\alpha t_0}.$$

Letting $t_0 \to \infty$ we get that we can approximate the systems with mutations by ones where mutation is turned off after t_0 uniformly in N. Then the claim follows by using the result for $O(1)$ initial mass of type 2 and no mutation.

3.5 Third moments

We expect that the random variables $\mathcal{W}^*({}^*\mathcal{W})$ have moments of all orders which determine the distribution. We do not verify this here but in this section we verify that the third moment of the rare mutant type is finite given if we start with total mass of type 1 only using the same method as that used for the first and second moments which for convenience we review here.

Lemma 3.32 *(First, second and third moments of $\mathcal{W}^*$)*

Consider times $t_N \to \infty$ with $t_N = o(\log N)$. The first, second and third moment of the rescaled total type two mass satisfies:

$$\lim_{N\to\infty} e^{-\alpha t_N} E[(\widehat{x}_2^N(t_N))] = m^* E[W], \tag{3.917}$$

$$\lim_{N\to\infty} e^{-2\alpha t_N} E[(\widehat{x}_2^N(t_N))^2] = (m^*)^2 (E[W])^2 + 2m^*\kappa_2 (E[W])^2 \tag{3.918}$$

and for some $0 < \kappa_3 < \infty$, $m^* = \frac{1}{c}(1+\frac{\gamma}{\alpha})m$ (as in (3.604)), κ_2 given by (3.788)

$$\lim_{N\to\infty} e^{-3\alpha t_N} E[(\widehat{x}_2^N(t_N))^3] = (m^*)^3 (E[W])^3 + 6(m^*)^2\kappa_2 (E[W])^3 + 6m^*\kappa_3 (E[W])^3. \tag{3.919}$$

□

Proof of Lemma 3.32

The proof will be based on duality and as a prerequisite we need (in order to determine the moment asymptotics) to evaluate the expression which represents the moments in terms of the dual particle system. We therefore carry out the proof in *two steps*, first an expansion of dual expression and then the moment calculation.

Step 1 *(Expansion of $\Pi^N(t)$ up to terms of order $O(\frac{1}{N^3})$)*

In this subsection we extend the coloured particle system in order to derive and expansion of $\Pi_t^{N,a}$ including terms of order $\frac{1}{N}$ and $\frac{1}{N^2}$ which are needed in the calculation of the second and third moments of $\widehat{x}_1^N(t)$.

Lemma 3.33 *(Expansion of $\Pi_{t_N}^{N,a}$)*

Consider the dual particle system starting with one, two or three particles. To obtain a unified expression we state the result using W_a.

The corresponding CMJ random growth constants (recall (3.143) $W_a = W_1$ with one initial particle, $W_a = W_1 + W_2$ if two particles start at different sites, $W_a = W^{(2)}$ if two particles start at the same site, $W_a = W_1 + W_2 + W_3$ is three particles start at different sites and $W_a = W_1^{(2)} + W_2$ if two particles start at one site and one particle at a different site.

Then there exists $\kappa_3 > 0$ such that

$$\Pi_{t_N}^{N,a} \sim \frac{1}{c}(\alpha+\gamma)\Big[(W_a)e^{\alpha t_N} - \frac{2\kappa_2}{N}(W_a^2)\frac{e^{2\alpha t_N}}{N} - \frac{3\kappa_3}{N^2}(W_a^3)e^{3\alpha t_N} + O(\frac{e^{4\alpha t_N}}{N^3})\Big]. \tag{3.920}$$

Then we get for the time-integral and the exponential of it:

$$H_a^N(t) := \frac{m}{N}\int_0^{t_N} \Pi_s^N ds \sim \frac{m^*}{N}(W_a)e^{\alpha t_N} - \frac{m^*\kappa_2}{N^2}(W_a^2)e^{2\alpha t_N} - \frac{m^*\kappa_3}{N^3}W_a^3 e^{3\alpha t_N} + O(\frac{e^{4\alpha t_N}}{N^4}), \tag{3.921}$$

$$e^{-H_a^N(t)} = 1 - H_a^N(t) + \frac{1}{2}(H_a^N(t))^2 - \frac{1}{6}(H_a^N(t))^3 + O((H_a^N(t))^4). \tag{3.922}$$

□

Proof of Lemma 3.33 We use an extension of the multicolour system defined so far in Subsubsection 3.2.9 (Step 1) therein and in the above Subsection 3.4 in order to identify $\Pi_{t_N}^{N,a}$ up to terms of order $e^{3\alpha t_N}$ with an error term of order $O(\frac{e^{4\alpha t_N}}{N^3})$. To obtain this we must introduce new colours.

Recall in the expansion to order $e^{2\alpha t}$ we worked with the system of white, black, red, green, purple and blue coloured particle system and that the dual system is given by the white, red and purple particles. The green particles are produced by coalescence of a red and white particle and a green site is produced by the migration of a green particle. These represent particles which are lost due to one collision (followed by coalescence). The white particles grow as $W_a \frac{\alpha+\gamma}{c} e^{\alpha t}$ and the green particles are produced at rate $W_a^2 e^{2\alpha s}$ at time s. Since the green families grow with exponential rate α the total number of green particles grows like $W_a^2 \kappa_2 e^{2\alpha t}$. The number of purple-blue particles is of order $O(e^{3\alpha t})$ and the the lower bound is given by the white and red particles obtained by subtracting the number of green particles. The upper bound is obtained by adding the blue particles.

In order to identify the *next order term* we must include the purple particles (and exclude the blue particles) and follow what happens when the purple particles coalesce or migrate. This requires to introduce two further colours, namely *pink* and *yellow*. When a purple particle migrates and hits a white, red or purple site we now produce a pink-yellow pair. There are $O(e^{4\alpha t})$ pink (yellow) particles where now the pink particles provide the dual and the yellow particles now behave as the blue particles did before in obtaining the error term.

We note that the purple particles can coalesce with a white (or red) particle and when this happens a white (or red) and green (to distinguish these we label them G_2) pair of particles is produced at the site where coalescence occurs. Green (G_2) particles are produced at rate $\text{const} \cdot W_a^3 e^{3\alpha t}$.

The two particle systems we have to compare, the dual one and the collision-free one, are now given as follows.

- The dual is given by the *white, red, purple* and *pink particles* and
- the collision free by *white, red, green* ($G \cup G_2$), *purple* and *yellow*.

The modified rules are now:

- If a purple migrates and hits a site occupied by white or red it dies and creates a pink-yellow pair which have subsequently coupled times of birth, migration,
- the yellow particle is placed in the first unoccupied site in the first copy of $\mathbb{N}$,
- if a purple particles coalesces with a white or red particle it produces a green G_2 particle
- the pink particles behaves like a standard dual particle.

This will allow us to approximate the dual with an error of order as: $N \to \infty$:

$$O(N^{-3} e^{4\alpha t_N}). \tag{3.923}$$

Now we have to represent the quantity $\Pi_t^{N,a}$ in terms of the numbers of particles of the various colours. Here we follow the procedure explained in Subsubsection 3.3.5, Part 2 therein. Then we obtain the claimed expansion of (3.920). Recall that $\kappa_2 W^2 e^{2\alpha t}$ denotes the term corresponding to the growth of the green G particles in subsubsection 3.3.5. In the same way the $\kappa_3 W^3 e^{3\alpha t}$ in (3.920) term corresponds to the set of G_2 particles, that is, the particles lost due to particles involved in exactly two collisions.

The assertions (3.921) and (3.922) follow from (3.920) by explicitly calculating the integral and then inserting the expansion in the Taylor expansion of the exponential around 0. q.e.d.

Step 2 *(Moment calculation)*

First recall the dual identity

$$E[(x_1^N(i,t))^k] = E\left[e^{-H_k^N(t)}\right], \tag{3.924}$$

where H_k^N is given by (3.921) and where W_a is the CMJ random growth constant (recall (3.143)) obtained by starting the dual with k particles at site i. We obtain a similar expression for $E[x_1^N(i,t)x_1^N(j,t)]$, where now we start one particle at i and one particle at j, etc.

We now compute the first three moments following the same method. Note that

$$\begin{aligned} E[(\widehat{x}_2^N(t))] &= E[N - \widehat{x}_1(t)] = N - NE[x_1(t)] \\ &= N - NE[(1 - m^*\frac{W_1}{N}e^{\alpha t} + O(\frac{1}{N^2}))]. \end{aligned} \tag{3.925}$$

Then replacing t by t_N we get

$$\lim_{N\to\infty} e^{-\alpha t_N} E[(\widehat{x}_2^N(t_N))] = \lim_{N\to\infty} e^{-\alpha t_N} N[\frac{m^* E[W]}{N} e^{\alpha t_N} + O(\frac{1}{N^2})] \tag{3.926}$$

and (1.19) follows provided that $t_N = o(\log N)$.

We now consider the second and third moments:

$$\begin{aligned} &E[(\widehat{x}_1^N(t))^2] = \\ &NE[(x_1^N(i,t))^2] + N(N-1)E[(x_1^N(i,t))x_1^N(j,t)], \quad i \neq j \end{aligned} \tag{3.927}$$

and for $i,j,k \in \{1,\cdots,N\}$ distinct

$$\begin{aligned} E[(\widehat{x}_1^N(t))^3] = \quad & NE[(x_1^N(i,t))^3] + 3N(N-1)E[(x_1^N(i,t))^2 x_1^N(j,t)] \\ & + N(N-1)(N-2)E[x_1^N(i,t)x_1^N(j,t)x_1^N(k,t)]. \end{aligned} \tag{3.928}$$

We first illustrate the method of calculation by calculating in detail the second moment. We substitute (3.921 and (3.922) in (3.933) keeping track of only those terms that do not go to 0 as $N \to \infty$. We get:

$$\begin{aligned} &E[(\widehat{x}_2^N(t))^2] = E[N - \widehat{x}_1^N(t)]^2 \\ &= N^2 - 2NE[\widehat{x}_1^N(t)] + E[\widehat{x}_1^N(t))^2] \\ &= N^2 - 2N^2E[x_1^N(t)] + NE[(x_1^N(t))^2] + N(N-1)E[x_1^N(i,t)x_1^N(j,t)]. \end{aligned} \tag{3.929}$$

We now insert for the appearing moments of x_1^N the formula (3.924) and then use the expansion of the exponential from (3.922). We first compute the terms coming from the $1 - H^N(t)$ in the expansion of the exponential. This gives

$$\begin{aligned} &N^2 - 2N^2 E\left[1 - \frac{m^*}{N}We^{\alpha t} + \frac{m^*\kappa_2}{N^2}W^2e^{2\alpha t} + O\left(\frac{e^{3\alpha t}}{N^3}\right)\right] \\ &+NE\left[1 - \frac{m^*}{N}W^{(2)}e^{\alpha t} + \frac{m\kappa_2}{N^2}W^2e^{2\alpha t} + O\left(\frac{e^{3\alpha t}}{N^3}\right)\right] \\ &+N(N-1)E\big[1 - \frac{m^*}{N}(W_1+W_2)e^{\alpha t} + \frac{m^*\kappa_2}{N^2}(W_1+W_2)^2e^{2\alpha t}\big] \\ &= N^2(1-2+1) + NE\left[(2m^*We^{\alpha t} + 1 - 1 - m^*(W_1+W_2)e^{\alpha t}) + O\left(\frac{e^{3\alpha t}}{N^3}\right)\right] \\ &E\left[-2m^*\kappa_2W^2e^{2\alpha t} - m^*W^{(2)}e^{\alpha t} + m^*\kappa_2(W_1+W_2)^2e^{2\alpha t} + O\left(\frac{e^{3\alpha t}}{N^3}\right)\right] \\ &= 2m^*\kappa_2(E[W])^2e^{2\alpha t} + m^*E[W_1+W_2]e^{\alpha t} - m^*E[W^{(2)}]e^{\alpha t} + O\left(\frac{e^{3\alpha t}}{N}\right). \end{aligned} \tag{3.930}$$

We next note that the $\frac{1}{2}(H^N(t))^2$ gives

$$(m^*)^2(E[W])^2e^{2\alpha t}. \tag{3.931}$$

Here W_1, W_2 are independent random variable coming from the CMJ limit starting with particles at two disjoint sites and $W^{(2)}$ is the random variable coming from the CMJ limit starting with two particles at the same site.

Combining all these terms we obtain

$$\begin{aligned} &E[(\widehat{x}_2^N(t))^2] \\ &= 2(E[W])^2 m\kappa_2 e^{2\alpha t} + 2m^*E[W]e^{\alpha t} - m^*E[W^{(2)}]e^{\alpha t} + (m^*)^2(E[W])^2e^{2\alpha t} \\ &+O\left(\frac{e^{3\alpha t}}{N}\right). \end{aligned} \tag{3.932}$$

Letting $t = t_N$, multiplying by both sides $e^{-2\alpha t_N}$ and taking the limit $N \to \infty$, we obtain (3.918) provided that $t_N = o(\log N)$.

We now follow the same method for the third moment but omit writing out the numerous intermediate terms which cancel and get:

$$\begin{aligned} &E[(\widehat{x}_2^N(t))^3] \\ &= N^3 - 3N^2E[\widehat{x}_1^N(t)] + 3NE[(\widehat{x}_1^N(t))^2] - E[(\widehat{x}_1^N(t))^3] \\ &= N^3 - 3N^3E[(x_1^N(t))] + 3N^2E[(x_1^N(t))^2] + 3N^2(N-1)E[x_1^N(i,t)x_1^N(j,t)] \\ &-NE[(x_1^N)^3] - 3N(N-1)E[(x_1^N(i))^2x_1(j)] - (N^3 - 3N^2 + 2N)E[x_1^N(i)x_1^N(j)x_1^N(k)]. \end{aligned} \tag{3.933}$$

The proof proceeds by substituting (3.921) and (3.922) via (3.924) in (3.933). One can verify by direct algebraic calculations that the resulting coefficients of N^3, N^2, N^1 are zero. The $O(1)$ terms involves powers of $e^{\alpha t}$, $e^{2\alpha t}$ and $e^{3\alpha t}$. To prove the Lemma it suffices to identify the coefficient of $e^{3\alpha t}$. The coefficient involving $(m^*)^3$ comes from the term $\frac{N^3}{6} \cdot (H_N(t))^3$ in the expansion of the exponential. The κ_2 term comes from the term $\frac{N^3}{2} \cdot (H_N(t)^2$ and the κ_3 term comes from the term $N^3H_N(t)$ in the expansion of the exponential.

To illustrate this structure we consider the contribution coming from the term

$$\frac{N^3}{2}(H_a^N)^2 = -(m^*)^2\kappa_2 e^{3\alpha t}(W_a)^3. \tag{3.934}$$

We must apply this to each term in the following expression

$$N^3[-3E(x_1^N(t)) + 3E[x_1^N(i,t)x_1^N(i,t)] - E[x_1^N(i,t)x_1^N(j,t)x_1^N(k,t)] \tag{3.935}$$

The corresponding terms with a $(W_a)^3$ result in

$$-3E[W_1^3]3E[(W_1+W_2)^3] - E[(W_1+W_2+W_3)^3] = -6E[W_1]E[W_2]E[W_3]. \tag{3.936}$$

Note that the coefficients of $e^{\alpha t}, e^{2\alpha t}$ depend on higher moments of W. This leads to

$$\begin{aligned} E[(\widehat{x}_2^N(t))^3] &= m^*\kappa_3(E[W])^3e^{3\alpha t} + 6(m^*)^2\kappa_2(E[W])^3e^{3\alpha t} + 6(m^*)^3(E[W])^3e^{3\alpha t} \\ &+\text{const} \cdot e^{2\alpha t} + \text{const} \cdot e^{\alpha t} + O\left(\frac{e^{\alpha 4t}}{N}\right) \end{aligned}$$

Letting $t = t_N$, multiplying by $e^{-3\alpha t_N}$ and taking the limit $N \to \infty$ we obtain (3.919) provided that $t_N = o(\log N)$.

3.6 Propagation of chaos: Proof of Proposition 1.15

Here we prove Proposition 1.15. We proceed in three steps.

Step 1 Fix $t_0 > -\infty$ and tagged sites $1, \dots, L$. We can again prove for $T_N = \frac{1}{\alpha} logN$ that $\{x_2^N(1, T_N + t_0), \dots, x_2^N(L, T_N + t_0)\}$ converges in distribution as $N \to \infty$ using the dual representation to show that all joint moments converge. Here of course it is equivalent to do the same for $x_1(i, T_N + t_0)$ which is more convenient for calculation. In particular we argue as below (3.595) using the dual process, except that now we start k_i-particles at site $i, i = 1, \dots, L$ to calculate the mixed higher moments of $x_1^N(1, T_N + t_0), \cdots, x_1^N(L, T_N + t_0)$, i.e.

$$E[\prod_{j=1}^{L} (x_1^N(j, T_N + t_0))^{k_j}], \quad k_1, k_2, \cdots, k_L \in \mathbb{N}, \tag{3.937}$$

in terms of

$$E[exp(-\frac{m}{N} \int_0^{T_N+t} \Pi_s^{N,(k_1,\cdots,k_L)} ds)] \tag{3.938}$$

using formula (3.716) and (3.717) and denoting with the superscript $(k_1, \cdots, k_L)$ the initial position.

Then we obtain convergence as $N \to \infty$ from Proposition 3.8 part (e). (Note the fact that the k_i here are possibly different does not change the argument). Therefore we have proved the convergence of the marginal distribution at time t_0 of the tagged size-L sample.

Step 2 Next we verify that the limiting dynamics for $t \geq t_0$ is as specified in (1.106). To do this we first review some results proved earlier. Using the results from Step 1 and Skorohod representation of weakly converging laws on a Polish space we can assume that $\Xi_N^{\log,\alpha}(t_0, 2) \underset{N\to\infty}{\longrightarrow} \mathcal{L}_{t_0}(2),\ a.s.$. Note that conditioned on $\mathcal{L}_{t_0}(2)$, the path $\{\mathcal{L}_t(2)\}_{t \geq t_0}$ is deterministic.

Consider for $i \in \{1, \dots, L\}$ the system:

$$\begin{aligned} dx_2^N(i,t) = c(\bar{x}_2^N(t) - x_2^N(i,t))dt \quad & - \quad s\, x_2^N(i,t)(1 - x_2^N(i,t))dt \\ & - \quad \frac{m}{N} x_2^N(i,t)dt \\ & + \quad \sqrt{d \cdot x_2^N(i,t)(1 - x_2^N(i,t))} dw_2(i,t). \end{aligned} \tag{3.939}$$

Conditioned on $\mathcal{L}_{t_0}(2)$, we know already for $N \to \infty$ (on our Skorohod probability space):

$$\bar{x}_2^N(t) \to \int x \mathcal{L}_t(2, dx),\ a.s., \quad \text{for } t \geq t_0. \tag{3.940}$$

Since we know from Step 1 that

$$\mathcal{L}[\{x_2^N(i, t_0),\ i = 1, 2, \cdots, L\}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\{x_2^\infty(i, t_0),\ i = 1, \dots, L\}], \tag{3.941}$$

then a standard coupling argument yields

$$\mathcal{L}[\{x_2^N(i,t)\}_{i=1,\dots,L;\ t\geq t_0}] \underset{N\to\infty}{\Longrightarrow} \mathcal{L}[\{x_2^\infty(i,t)_{i=1,\dots,L;\ t\geq t_0}\}], \tag{3.942}$$

where $x_2^\infty(\cdot, \cdot)$ satisfies for $i = 1, \dots L, \quad t \geq t_0$.

$$\begin{aligned} dx_2^\infty(i,t) = \quad & c(\int x \mathcal{L}_t(t, dx) - x_2^\infty(i,t))dt - s\, x_2^\infty(i,t)(1 - x_2^\infty(i,t))dt \\ & + \sqrt{d \cdot x_2^\infty(i,t)(1 - x_2^\infty(i,t))} dw_2(i,t). \end{aligned} \tag{3.943}$$

Step 3 Now we need to show that we have for the equation (3.943) a unique solution for $t \in \mathbb{R}$.

We can prove that, given $(\int x\mathcal{L}_t(2,dx))_{t\in\mathbb{R}}$, which converges to 0 as $t \to -\infty$, (3.943) has a unique solution on $(-\infty, t_0]$ as follows.

We can construct a minimal solution by considering a sequence (in the parameter n) of solutions corresponding to starting the process at time t_n. Then consider the tagged site SDE driven by this mean curve starting at 0 at time $t_n \to -\infty$. The sequence of solutions forms a stochastically monotone increasing sequence as $n \to \infty$ (use coupling). Then observe that any solution $x(t)$ that satisfies $x(t) \to 0$ as $t \to -\infty$ must have zeros (cf. classical result on Wright-Fisher diffusions). Then by coupling at a zero we get that it must agree with the minimal solution. This gives uniqueness for a given mean path $(\mathcal{L}_t(2))_{t\in\mathbb{R}}$. Note that this also implies the uniqueness of $(\mathcal{L}_t)_{t\in\mathbb{R}}$ since we have already established the uniqueness of the limiting mean curve.

This completes the proof.

3.7 Extensions: non-critical migration, selection and mutation rates

The hierarchical mean field analysis allows us to investigate the emergence times and behaviour for a wide range of scenarios involving different parameter ranges for mutation rates, migration rates and relative fitness of the different levels. Our main focus in this work is the *critical case* in which mutation, migration and fitness all play a comparable role and for this reason we have chosen the parameterization $\beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 1$. However the basic tools and analysis we develop here can be adapted to other scenarios. Although we will not carry out the analysis in detail we now briefly indicate the main features of the different cases.

For example, consider the following "general" parametrization of the migration, selection and mutation rates:

$$c_{N,k} = \frac{c}{N^{(k-1)(1-\beta_1)}},\quad 0 \le \beta_1 < 1, \tag{3.944}$$

$$s_N = \frac{s}{N^{\beta_2}},\quad \beta_2 \ge 0, \tag{3.945}$$

$$m_N = \frac{m}{N^{\beta_3}},\quad \beta_3 \ge 0. \tag{3.946}$$

The different ranges of the migration parameters have the following interpretation. The value $\beta_1 = 0$ corresponds to euclidian space with dimension 2. The values $0 < \beta_1 < 1$ correspond to dimensions $d > 2$. We do not consider this case in this paper but for a detailed discussion of the relation of random walks on the hierarchical group to random walks on $\mathbb{Z}^d$ and for further references see [DGW01].

We have discussed the case $\beta_1 = \beta_2 = 0$ and $\beta_3 = 1$ above and we will now briefly discuss the case (1) where $\beta_1 = 0$ but $\beta_2 > 0$ and (2) the cases $\beta_1 = \beta_2 = 0$ but either $\beta_3 < 1$ or $\beta_3 > 1$, which are the other most interesting cases.

(1) In order to understand the behaviour $\beta_1 = 0,\ \beta_2 > 0$ consider the behaviour of the dual process. Then the dual process has the property that most occupied sites have only one factor (until the number of factors is $O(N)$) and the number of factors grows like $e^{\frac{st}{N^{\beta_2}}}$. In speeded-up time scale $N^{\beta_2}t$ the factors move quickly with possibility of coalescence each time a pair collides. If $\beta_3 \le 1$, then heuristically, emergence will occur when

$$e^{\frac{st}{N^{\beta_2}}} = O(N^{\beta_3}) \tag{3.947}$$

that is,

$$t = O(N^{\beta_2}\frac{\beta_3}{s}\log N). \tag{3.948}$$

This means that emergence occurs much later than in the critical regime considered in our work.

(2) We now restrict our attention to the case $\beta_1 = \beta_2 = 0$ and discuss the effect of different mutation rates. The case of *faster mutation*, $0 < \beta_3 < 1$, has the effect that emergence can occur before collisions become non-negligible. In particular, if the mutation rate is $\frac{m}{N^{\beta_3}}$ with $\beta_3 < 1$ we get emergence at time $\beta_3\alpha \log N$ and the *emergence is deterministic*, i.e. $\mathrm{Var}(\bar{x}_2^N(\alpha\beta_3 \log N)) \to 0$. In this case if we look at the dual process there are $O(N^{1-\beta_3})$ occupied sites at time $O(1)$ and therefore we have this number of droplets of size N^{β_3} at time $\alpha\beta_3 \log N$. Therefore we have a law of large numbers and deterministic emergence dynamics.

On the other hand, *slower mutation*, $\beta_3 > 1$, means that the "wave of advance" of the dual system must move to distance 2 or higher depending on the value of $\beta_3 > 1$ in order to produce the number of dual particles needed to guarantee emergence.

4 Appendix. Nonlinear semigroup perturbations

We use the following result of Marsden [MA], (4.17).

Theorem 1 *(Perturbation)*

Let $\mathbb{B}$ be a Banach space and let A_S be the infinitesimal generator of a strongly continuous semigroup*, with $\|S_t\| \leq Me^{Ct}$ for some C. Let $A_T : \mathbb{B} \to \mathbb{B}$ be a vector field on $\mathbb{B}$ such that A_T is of class C^2 with its first and second derivatives uniformly bounded on bounded subsets and let $\{T_t\}$ be the flow of A_T.*

Then $A_S + A_T$ has a unique flow which is Lipschitz for each t, $0 \leq t \leq T$, and

$$V_t x = \lim_{n\to\infty} (S_{t/n} \cdot T_{t/n})^n x \tag{4.1}$$

uniformly in t for each x on bounded sets of t. If $x \in \mathcal{D}(A_S + A_T)$, then

$$\frac{d}{dt} V_t x = (A_S + A_T) U_t \tag{4.2}$$

on $[0, \tau)$ where τ is the exit time from $\mathbb{B}$. □

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